MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

State Meet - March 27, 2025 Round 1 - Arithmetic and Number Theory

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Compute $53^2 + 53 \cdot 47 + 47^2$.

Solution:

$$= (53 + 47)^2 - 53 \cdot 47$$
$$= 100^2 - (50 + 3)(50 - 3)$$
$$= 100^2 - 50^2 + 3^2$$
$$= 7509$$

2. Find the greatest common divisor of 30461 and 56689.

Solution: We denote (a, b) as the GCD of a and b. Using the Euclidean Algorithm, we have (30461, 56689) = (30461, 56689 - 30461) $= (30461, 26228) = (4233, 26228) = (4233, 830) = \boxed{83}$

3. Find the sum of the three 4-digit numbers \overline{ABCD} divisible by 13 such that 2A - 1 = D and B + 1 = C. Note that $A \neq 0$.

Solution: We have modulo 13 that

 $\overline{ABCD} = 1000A + 100B + 10C + D$

 $\equiv (1001 - 1)A + (104 - 4)B + (13 - 3)C + D$

$$\equiv -A - 4B - 3C + D$$
$$= -A - 4B - 3(B+1) + 2A - 1$$
$$= A - 7B - 4$$
$$= A + 6B - 4$$

The possible values (A, B) such that $A + 6B \equiv 4 \pmod{13}$ can be found by plugging in values of B and solving for A. This gives tuples (4, 0, 1, 7), (11, 1, 2, 21), (5, 2, 3, 9), (12, 3, 4, 23), (6, 4, 5, 11), (0, 5, 6, -1), (7, 6, 7, 13), (1, 7, 8, 1), (8, 8, 9, 15), (2, 9, 10, 3). The valid 4-digit numbers are then 4017, 5239, and 1781. Their sum is 11037.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Round 2 - Algebra I

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Solve for x. Express your answer as a fraction.

$$8^x = 4^{-\frac{3}{2}} \cdot 16^{27^{-\frac{1}{3}}}$$

Solution:

$$\Rightarrow 2^{3x} = 2^{-3}2^{4 \cdot \frac{1}{3}}$$
$$\Rightarrow 3x = -3 + \frac{4}{3} = -\frac{5}{3}$$
$$\Rightarrow x = \boxed{-\frac{5}{9}}$$

2. Joey takes x minutes to eat 8 burgers. Emily takes 2 minutes to eat 12 burgers. If Joey and Emily combined take x^2 minutes to eat 350 burgers, solve for x.

Solution: Joey eats $\frac{8}{x}$ burgers a minute. Emily eats 12/2 = 6 burgers a minute. In x^2 minutes they eat $x^2(\frac{8}{x}+6) = 8x + 6x^2$ burgers. Then,

$$8x + 6x^2 = 350$$
$$\Rightarrow 3x^2 + 4x - 175 = 0$$
$$\Rightarrow (3x + 25)(x - 7)$$

Since x is positive, we have $x = \boxed{7}$.

3. Let a be the largest real solution to

$$(x^{3} + \frac{1}{x^{3}}) - 3(x^{2} + \frac{1}{x^{2}}) + 5(x + \frac{1}{x}) - 12 = 0$$

Find $a - \frac{1}{a}$.

Solution:

We know $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$ and $x^3 + \frac{1}{x^3} = (x + \frac{1}{x})^3 - 3(x + \frac{1}{x})$. Then, letting $y = x + \frac{1}{x}$, we have

$$y^{3} - 3y - 3(y^{2} - 2) + 5y - 12$$

 $\Rightarrow y^{3} - 3y^{2} + 2y - 6$
 $\Rightarrow (y - 3)(y^{2} + 2)$

Since x is real, y = 3, so $a + \frac{1}{a} = 3$. Then,

$$a - \frac{1}{a} = \sqrt{a^2 - 2 + \frac{1}{a^2}} = \sqrt{(a + \frac{1}{a})^2 - 4} = \sqrt{5}$$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Round 3 - Geometry

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Regular octagon ABCDEFGH has side length 2. What is the area of the square ACEG?



Solution: Extend *AB* and *DC* to meet at *X*. We have ΔXBC is a isosceles right triangle, so $XB = XC = \frac{2}{\sqrt{2}} = \sqrt{2}$. Using the Pythagorean Theorem $AC^2 = AX^2 + CX^2 = (2 + \sqrt{2})^2 + (\sqrt{2})^2 = \boxed{8 + 4\sqrt{2}}$

2. Three circles with radii 5, 5, and 8 are all externally tangent to each other. Compute the radius of the circle that circumscribes all three circles. Express your answer as a fraction.



Solution: Let the radius be r. Let the center of the circle with radius 8 be A, and the centers of the circles with radius 5 be B and C. Extend the lines connecting the points of tangency with the circumscribing circle to the centers of the smaller circles, to meet at the center of the circumscribing circle, O. Draw altitude AD of ΔABC . We have AB = AC = 5 + 8 = 13 and BD = CD = 5, so AD = 12 by the Pythagorean Theorem. Also we have AO = r - 8 and BO = r - 5, so DO = AD - AO = 20 - r. By the Pythagorean Theorem on ΔBDO , we have



3. Points A, B, C, D lie on a circle. Suppose AC and BD intersect inside the circle at point P, and rays AB and DC intersect outside the circle at point Q. If AP = 2, DP = 5, BQ = 8, CQ = 6, find AB + DC.



Solution: Let x = AB. Since $\triangle ABP \sim \triangle DCP$, we have $AB/DC = AP/DP = \frac{2}{5}$, so $DC = \frac{5}{2}x$. By Power of a Point on Q, we have $QB \cdot QA = QC \cdot QD$ $\Rightarrow 8(8 + x) = 6(6 + \frac{5}{2}x)$ $\Rightarrow 64 + 8x = 36 + 15x$ $\Rightarrow 7x = 28$ $\Rightarrow AB = 4, DC = 10$ Thus, $AB + DC = \boxed{14}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Round 4 - Algebra II

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Given that x is a positive integer, solve for x.

$$x = 2\log_2(2x)$$

Solution:

$$\Rightarrow 2^{x} = 2^{2 \log_{2}(2x)}$$
$$\Rightarrow 2^{x} = (2x)^{2} = 4x^{2}$$
$$\Rightarrow 2^{x-2} = x^{2}$$

Since x is an integer, x must be a power of 2. Trying small powers of 2, we see that x = |8|.

2. Consider functions f(x) = ax + 3, g(x) = 2x + b where a, b are positive integers. Determine a + b, given that f(g(x)) = g(f(x)).

Solution: We have f(g(x)) = a(2x+b)+3 = 2ax+ab+3 and g(f(x)) = 2(ax+3)+b = 2ax+b+6. Thus,

$$ab + 3 = b + 6$$
$$ab - b = 3$$
$$(a - 1)b = 3$$

Then, since a, b are positive integers, a-1 and b must be 1 and 3, in some order. Then $a-1+b=1+3 \Rightarrow a+b=5$.

3. Given real x, y such that

$$x(1 + \frac{1}{x^2 + y^2}) = 5$$

$$y(1 - \frac{1}{x^2 + y^2}) = 7$$

compute $z^2 - (5+7i)z$ where z = x + yi.

Solution: Adding the first equation plus i times the second equation, we get

$$x + yi + \frac{x - yi}{x^2 + y^2} = 5 + 7i$$

Since $x^2 + y^2 = (x - yi)(x + yi)$, we have

$$x + yi + \frac{1}{x + yi} = 5 + 7i$$

$$\Rightarrow z + \frac{1}{z} = 5 + 7i$$

$$\Rightarrow z^{2} + 1 = (5 + 7i)z$$

$$\Rightarrow z^{2} - (5 + 7i)z = \boxed{-1}$$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Round 5 - Analytic Geometry

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Consider the equation of a circle below. Find the length of a line segment connecting the origin to a point on the circle such that the line segment is tangent to the circle.

$$x^2 - 4x + y^2 + 4y + 7 = 0$$

Solution: Rearranging, we get

$$(x-2)^2 + (y+2)^2 = 1$$

This is a circle centered at (2, -2) with radius 1. The center is $2\sqrt{2}$ away from the origin. By the Pythagorean Theorem, the length of the desired line segment is $\sqrt{(2\sqrt{2})^2 - 1^2} = \sqrt{7}$

2. The graph of the equation below circumscribes a rectangle whose vertical height is double its horizontal width. Find the area of the rectangle. Express your answer as a simplified fraction.

$$4(x-5)^2 + 9(y+2)^2 = 36$$

Solution: The area of the rectangle is the same as that of the rectangle with height double its width and circumscribed by the graph of ellipse

$$4x^2 + 9y^2 = 36$$

The top right corner lies on the line y = 2x, so the x-coordinate of this point satisfies

$$4x^2 + 9(2x)^2 = 36$$
$$\Rightarrow 10(2x)^2 = 36$$

$$\Rightarrow 2x = \frac{6}{\sqrt{10}}$$
$$\Rightarrow 2y = \frac{12}{\sqrt{10}}$$

The width and height of the rectangle are 2x and 2y, so the area is

$$\frac{6}{\sqrt{10}}\frac{12}{\sqrt{10}} = \frac{72}{10} = \boxed{\frac{36}{5}}$$

3. Find the area of the region whose vertices are the 4 points at which the graph of the equation below intersects the boundary of the square centered at the origin whose sides are of length 12 and parallel to the axes. Express your answer as a simplified fraction.

$$y = \frac{1}{18}(x^2 - 3x - 108)$$

Solution: The boundary of the square is formed by lines $x = \pm 6, y = \pm 6$. We plug in $x = \pm 6$ to get intersections (6, -5) and (-6, -3). The intersections with y = 6 satisfy

$$6 = \frac{1}{18}(x^2 - 3x - 108)$$
$$108 = x^2 - 3x - 108$$
$$x^2 - 3x - 216 = 0$$

which has solutions close to $\sqrt{216}$ which will clearly not be within -6 and 6. Thus, it does not intersect the square on the y = 6 line. The intersections with y = -6 satisfy

$$-6 = \frac{1}{18}(x^2 - 3x - 108)$$
$$-108 = x^2 - 3x - 108$$
$$x(x - 3) = 0$$

so this gives intersections (0, -6) and (3, -6).

We compute the area of the trapezoid with vertices (-6, -3), (-6, -6), (6, -6), (6, -5) minus the areas of the triangles with vertices (-6, -3), (-6, 6), (0, -6) and (6, -6), (6, -5), (3, -6), which is

$$24 - 9 - \frac{3}{2} = \frac{27}{2}.$$

Alternatively, you could use the Shoelace Theorem to calculate the area.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Round 6 - Trigonometry and Complex Numbers

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. Simplify as much as possible.

$$\frac{\cos(x)\sin(2x)}{\sin(x) - \cos^2(x) + 1} - 2$$

Solution: $= \frac{2\sin(x)\cos^{2}(x)}{\sin(x) + \sin^{2}(x)} - 2$ $= \frac{2\sin(x)(1 - \sin^{2}(x))}{\sin(x)(1 + \sin(x))} - 2$ $= 2(1 - \sin(x)) - 2 = \boxed{-2\sin(x)}$

2. Compute the expression below. Express your answer in rectangular form.

$$e^{i\frac{2\pi}{6}}(2+3i)\left(\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right)$$

Solution:

$$= (2+3i)e^{i\frac{2\pi}{6}}e^{-i\frac{5\pi}{6}}$$
$$= (2+3i)e^{-i\frac{\pi}{2}}$$

The $e^{-i\frac{\pi}{2}}$ just rotates 2+3i clockwise by $\frac{\pi}{2}$, which gives an answer of 3-2i

3. Find the number of complex solutions to

$$z^{51} + z^{52} + \dots + z^{2024} = 0$$

Solution:

$$\Rightarrow z^{51}(1 + z + \dots z^{1973}) = 0$$
$$\Rightarrow z^{51} \frac{z^{1974} - 1}{z - 1}$$

The solutions to this are 0 and the 1974th roots of unity, excluding 1. This gives $\boxed{1974}$ solutions.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES State Meet - March 27, 2025 Team Round

All answers must be in simplest exact form in the answer section. NO CALCULATORS ALLOWED

1. How many paths are there from (0,0) to (5,5) such that each move is one of the following 3 options: one step right, one step up, or two steps right and one step up.

Solution: Let f(x, y) be the number of paths from (0, 0) to (x, y). Then,

$$f(x,y) = f(x-1,y) + f(x,y-1) + f(x-2,y-1)$$

Starting from f(0,0) = 1 and f(x,y) = 0 if x < 0 or y < 0, we can inductively compute f(x,y) for every values $0 \le x, y \le 5$, producing this table:

| 1 | 6 | 26 | 86 | 241 | 592 |
|---|---|----|----|-----|-----|
| 1 | 5 | 19 | 55 | 136 | 296 |
| 1 | 4 | 13 | 32 | 68 | 128 |
| 1 | 3 | 8 | 16 | 28 | 44 |
| 1 | 2 | 4 | 6 | 8 | 10 |
| 1 | 1 | 1 | 1 | 1 | 1 |

For example, 68 is the sum of the values in the highlighted cells. Thus, the answer is 592.

2. Consider the sequence of integers defined as $a_1 = 5$, $a_{n+1} = 3n^2 - 2a_n$. Compute $a_1 + a_2 + \cdots + a_{12}$ given that $a_{13} = 19730$. For reference, the formula for the sum of the first *n* squares is

$$1^{2} + 2^{2} + \dots n^{2} = \frac{n(n+1)(2n+1)}{6}$$

Solution: Let $s = \sum_{i=1}^{12} a_i$. Then,

$$s - a_1 + a_{13} = \sum_{i=1}^{12} a_{i+1}$$
$$= \sum_{i=1}^{12} 3i^2 - 2a_i$$
$$= 3\frac{12(12+1)(2 \cdot 12+1)}{6} - 2s$$
$$= 1950 - 2s$$
$$\Rightarrow s - 5 + 19730 = 1950 - 2s$$
$$\Rightarrow 3s = -17775$$
$$\Rightarrow s = -5925$$

3. How many integers $n \ge 3$ is it possible that the angles of an *n*-gon form an arithmetic sequence with a constant difference of n - 3 degrees?

Solution: The average of the angles must be $\frac{180(n-2)}{n}$, and the difference between the largest and smallest angle is (n-1)(n-3). Thus, the smallest angle is $\frac{180(n-2)}{n} - \frac{(n-1)(n-3)}{2}$, which we need to be positive. Thus, we want

360(n-2) - n(n-1)(n-3) > 0

Estimating the polynomial as $360n - n^3$, we get that the maximal value of n should be around 19. We see that

$$360 \cdot 18 - 20 \cdot 19 \cdot 17 = 20(18 \cdot 18 - 19 \cdot 17) > 0$$

and

$$360 \cdot 19 - 21 \cdot 20 \cdot 18 = 20(18 \cdot 19 - 21 \cdot 18) < 0$$

. Thus, the largest possible n is 20, giving us 20 - 3 + 1 = 18 possible values of n.

4. Given distinct positive integers a, b, c that satisfy

$$(a!)(b!)^2(c!)^3 = 2^{14}3^55^3$$

compute a + 2b + 3c.

Solution: We note that a, b, c < 7 since the product has no powers of 7. To get 3 powers of 5, we must have that either c = 5 or c = 6 or a and b are 5 and 6 in some order. Case 1 c = 5: We have $(5!)^3 = 2^9 3^3 5^3$, so $(a!)(b!)^2 = 2^5 3^2$. Each factorial provides at most one power of 3, so b must be 3 or 4, neither of which give a solution. Case 2 c = 6: We have $(6!)^3 = 2^{12} 3^6 5^3$, which is too many powers of 3. Case 3 (a, b) = (6, 5): We have $(6!)(5!)^2 = 2^{10} 3^4 5^3$, so $(c!)^3 = 2^4 3$, which is not possible. Case 4 (a, b) = (5, 6): We have $(5!)(6!)^2 = 2^{11} 3^5 5^3$, so $(c!)^3 = 2^3$, giving solution (a, b, c) = (5, 6, 2). Then, $a + 2b + 3c = \boxed{23}$.

5. On the coordinate place, consider the L-shaped region of area 4 on the plane composed of the unit squares with lower-left corners at (0,0), (1,0), (2,0), and (0,1). By making cuts along two lines that intersect at the origin, this region is divided into 3 regions of equal area. Compute the sum of the slopes of these two lines. Express your answer as a fraction.

Solution: The cut with smaller slope intersects the region again on line x = 3 at a point that forms a triangle with points (0,0), (3,0) and an area equal to 4/3. This point (3,y) then must satisfy $\frac{3y}{2} = \frac{4}{3} \Rightarrow y = \frac{8}{9}$. This gives a slope of $\frac{8}{27}$.

The cut with larger slope intersects the region again on line x = 1 at a point that forms a trapezoid with points (0,0), (0,2), (1,2) and an area equal to 4/3. This point (1,y) then must satisfy $1 \cdot \frac{2+(2-y)}{2} = \frac{4}{3} \Rightarrow y = \frac{4}{3}$. This gives a slope of $\frac{4}{3}$.

Adding these slopes, we get $\left|\frac{44}{27}\right|$



6. Find the number of values of x between 0 and 10, inclusive, that satisfy

$$\sin(x^2\pi) = \cos(x\pi)$$

Solution: Each time $\sin(x^2\pi)$ goes from -1 to 1 or from 1 to -1, it intersect $\cos(x\pi)$ once. From x = 0 to the first time $\sin(x^2\pi) = 1$ at $x = \sqrt{0.5}$, it intersects $\cos(x\pi)$ once since $\cos(x\pi)$ is going from 1 to less than 0. Then, there is one intersection between $\sqrt{0.5} < x < \sqrt{1.5}$, one between $\sqrt{1.5} < x < \sqrt{2.5}$, all the way to an intersection between $\sqrt{98.5} < x < \sqrt{99.5}$. There is no intersection between $\sqrt{99.5} < x < 10$ since $\cos(x\pi)$ is near 1 there while $\sin(x^2\pi)$ is coming up from -1 to 0. Thus, there are 100 total intersections.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

State Meet - March 27, 2025 Team Round Answer Sheet

$\underline{\mathbf{ANSWERS}}$

- 8. _____

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

State Meet - March 27, 2025

Answer Key

| | Round 1 - Arithmetic and Number Theory | | Round 5 - Analytic Geometry |
|---------------|--|----|--|
| 1. | 7509 | 1. | $\sqrt{7}$ |
| 2. | 83 | 2. | $\frac{36}{5}$ |
| 3. | 11037 | 3. | $\frac{27}{2}$ |
| | Round 2 - Algebra I | | Round 6 - Trigonometry and Complex Numbers |
| $\frac{1}{2}$ | $-\frac{5}{9}$ 7 | 1. | $-2\sin(x)$ |
| 2. 3. | $\sqrt{5}$ | 2. | 3-2i |
| | Round 3 - Geometry | 3. | 1974 |
| 1. | $\frac{3}{8+4\sqrt{2}}$ | | Team Round |
| 2. | $\frac{40}{3}$ | 1. | 592 |
| 3. | 14 | 2. | -5925 |
| | Round 4 - Algebra II | 3. | 18 |
| 1. | 8 | 4. | 23 |
| 2. | 5 | 5. | $\frac{44}{27}$ |
| 3. | -1 | 6. | 100 |

School:

Circle your team below.

| Team $\#1$ | Team $#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|------------|-----------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name: _

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

- No two students from the same school should be sitting at the same table or in adjacent desks in a classroom.
- All electronic devices should be OFF and not in plain sight.
- Calculators are *not* allowed during this round.
- Do not turn this paper over until the moderator says, "Begin!"

MAML State Meet Round 1: Arithmetic and Number Theory

| Team $\#1$ | Team $#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|------------|-----------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name:

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

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MAML State Meet Round 2: Algebra I

| Team $\#1$ | Team $\#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|------------|------------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name:

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

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- Do not turn this paper over until the moderator says, "Begin!"

MAML State Meet Round 3: Geometry

| Team $\#1$ | Team $#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|------------|-----------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name:

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

- No two students from the same school should be sitting at the same table or in adjacent desks in a classroom.
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- Calculators are *not* allowed during this round.
- Do not turn this paper over until the moderator says, "Begin!"

MAML State Meet Round 4: Algebra II

| Team $\#1$ | Team $\#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|------------|------------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name: ____

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

- No two students from the same school should be sitting at the same table or in adjacent desks in a classroom.
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- Calculators are *not* allowed during this round.
- Do not turn this paper over until the moderator says, "Begin!"

MAML State Meet Round 5: Analytic Geometry

| Team #1 | Team $#2$ | Team $#3$ | Team $#4$ | Team $\#5$ |
|---------|-----------|-----------|-----------|------------|
| Team #6 | Team #7 | Team #8 | Team #9 | Team #10 |

Name: ____

For Official Use Only Score:

YES Western Mass ARML Member? NO (circle one)

Reminders:

- No two students from the same school should be sitting at the same table or in adjacent desks in a classroom.
- All electronic devices should be OFF and not in plain sight.
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MAML State Meet Round 6: Trigonometry and Complex Numbers

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

State Meet - March 27, 2025

TEAM ROUND

School Name:

Team #: _____

Team Members:

Score = ____ / 12