MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Round 1 **Arithmetic and Number Theory**

1. ____________________ %

2. ______________________

3. ______________________

1. Last week a bakery sold a loaf of bread weighing 4.5 pounds for \$6.00. Keeping the size the same, the bakery changed the ingredients so that the loaf weighed 3 pounds. Then it increased the price to \$7.50. What is the percent increase in the cost per pound of the bread?

2. Given that $p^2 - 12p + 35$ is prime, find <u>all</u> integer values of p.

3. Find all ordered pairs (A, B) such that when the two-digit number AA is divided by 75, the result is the two-digit non-zero decimal $0.BB$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Round 2 **Algebra 1**

1. ____________________

2. (______ , ______ , ______)

3. ____________________

1. Sam bought n items at \$$k$/item. He sold some of them for a profit of p% each.
He must sell a minimum of Q items to cover his cost, if Q is an integer and $[Q]+1$ items, if Q is not an integer. Determine a simplified expression for Q in terms of n and p.
Note: $[x]$ denotes the greatest integer in x, i.e. the largest integer less than or equal to x.

2. The real solution to $\sqrt{x}+28\sqrt[4]{x}=2013$ can be written as $(a \cdot b)^c$ where a and b are prime positive integers with $a<b$. Find the ordered triple (a, b, c).

3. Determine the number of ordered pairs (a, b) of positive integers a and b such that $\dfrac{2013}{a-b}=\dfrac{a+b}{2013}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

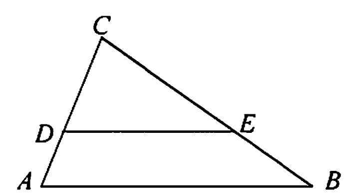
Round 3 – Geometry

1. ____________________

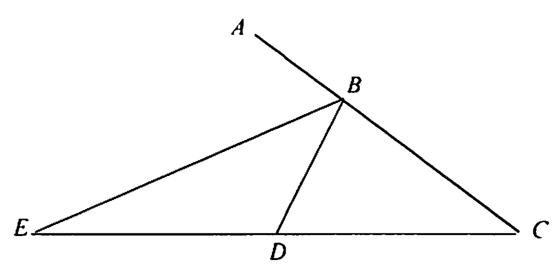
2. ____________________

3. ____________________

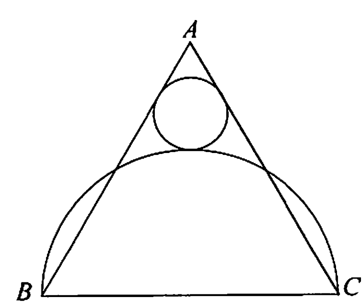
1. If $\overline{DE} \parallel \overline{AB}$, $AD = 4$, and the area of ΔCDE equals the area of $ABED$, compute the length of $\overline{CD}$.



2. Points A, B, and C are collinear.
ΔBDC is acute.
$\overline{CD}$ is extended to E so that angle $m\angle ABE = m\angle DBE$. Also, $m\angle C = m\angle E$.
Let $k = m\angle E$. Compute the largest possible integer value of k.
Note: The diagram is <u>not</u> drawn to scale.



3. A semicircle is drawn on the base of equilateral triangle ABC of side 2. A circle is inscribed in the triangle externally tangent to the semicircle. Compute the radius of the small circle.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Round 4 – Algebra 2

1. ______________________

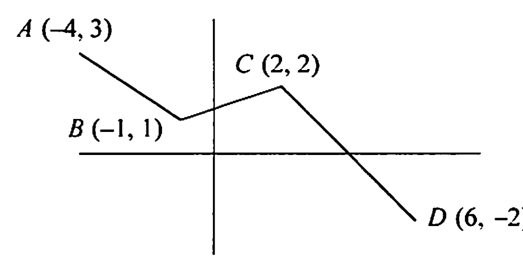
2. ______________

3. ______________________

1. Factor completely: $3 \cdot 4^x + 6 \cdot 2^x y - 45y^2$

2. Shown to the right is the complete graph of $y = f(x)$ from point A to point D. Find the value of the largest zero of $y = -3f(x+2) + 4$.

   Note: $x = a$ is a zero of $y = f(x) \Leftrightarrow f(a) = 0$.



3. Given the system $3x + y + 2z = 15$ and $x + y + z = 8$, determine all ordered triples, (x, y, z), which are the solutions of the system over the set of positive integers.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Round 5 – Analytic Geometry

1. _____

2. ______________________

3. ______________________

1. For $m < 0$ and $b > 0$, the area enclosed by $f(x) = mx + b$ and the positive x- and y-axes is 13. Compute the area enclosed by the positive x- and y-axes and $y = 2f\left(\frac{x}{2}\right)$.

2. A line is drawn through the origin passing through squares A, B, and C. The coordinates of the vertices of A are (0, 0), (2, 0), (2, 2), and (0, 2). For B they are (3, 1), (5, 1), (5, 3), and (3, 3) and for C they are (7, 2), (9, 2), (9, 4), and (7, 4). Determine the slope of the line such that the area of the regions below the line and inside the squares equals one-third of the total area of the squares.

3. The chord of circle $x^2 + y^2 = 25$ connecting $A(0, 5)$ and $B(4, 3)$ is the base of an equilateral triangle with the third vertex C in the 1st quadrant.
Compute the smallest possible x-coordinate of C.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Round 6 – Trig and Complex Numbers

1. ______________________

2. ______________________

3. _______

1. Determine all value(s) of x, $0° \le x° \le 180°$ such that $1 + \cos 2x = 2\sin(90° - x)$

2. There are two possible triangles ΔABC with $m\angle A = 60°, a = 7$, and $c = 8$.

   Determine the ratio of the areas of the two triangles determined by the data, larger to smaller.

3. Determine the <u>largest</u> solution to $\sin\frac{3x}{2} - \sin\frac{x}{2} = 0$ that lies in the half-open interval $\left[2013\pi,\ 2015\pi\right)$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013

Team Round

1. ____________________ **4.** ____________________

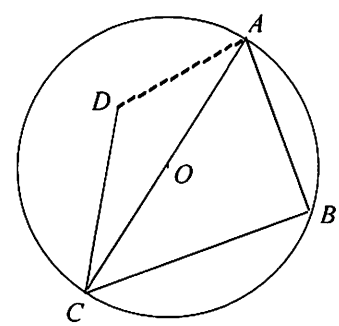
2. ____________________ **5.** ____________________

3. ____________________ **6.** ____________________

1. For each of the 900 integers in {100, 101, 102, . . . , 998, 999} whose ten's digit is nonzero, a quadratic equation is created by using the hundred's digit for the coefficient of x^2, the ten's digit for the coefficient of x, and the unit's digit for the constant. How many of those equations have a double root?

2. Let $f(x) = mx + b$ for $m \neq 0$ and $b > 4$. If $f(f(f(2))) = f(2)$, compute the <u>largest</u> possible value of m.

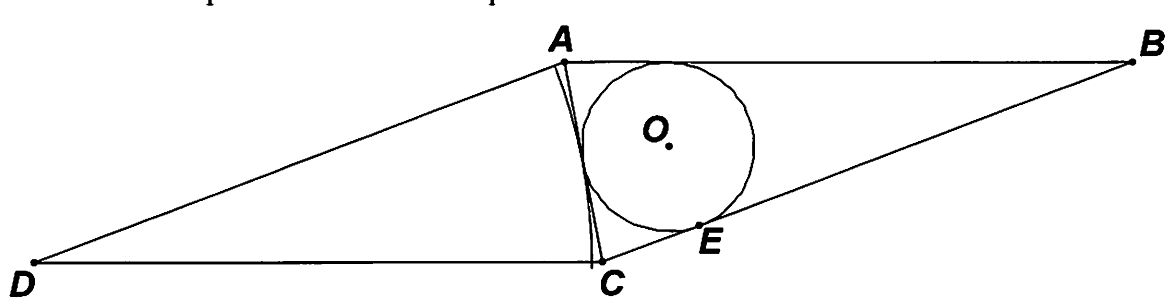
3. The length of diameter $\overline{AC}$ in circle O is 20. Also, $m\angle DCB = 60°$ and $AB = CD = 12$. The length of $\overline{AD}$ can be written in simplest terms as $a\sqrt{b + c\sqrt{3}}$. Find the ordered triple (a, b, c).



4. The numbers 2, 3, 4, 5, 6, 7, 8, 9, 10 are arranged at random. Compute the probability that at least one number is flanked by its factors, one on each side.

5. Compute the maximum value of $xy(120 - 5x - 4y)$ for positive values of x and y.

6. $ABCD$ is a rhombus, $AB = 6$ and $AC = 2$. Circle O is inscribed in ΔABC and circle D is tangent to $\overline{AC}$. Computee the value of the product of the radii of circles O and D.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2013 – ANSWER SHEET

Round 1

1. 87.5%
2. 4, 8
3. $(3, 4)$, $(6, 8)$

Round 2

1. $\frac{100n}{100+p}$
2. $(3, 11, 4)$
3. 13

Round 3

1. $4(\sqrt{2}+1)$
2. 29
3. $\frac{\sqrt{3}-1}{3}$

Round 4

1. $3(2^x - 3y)(2^x + 5y)$
2. $\frac{2}{3}$
3. $(1, 2, 5), (2, 3, 3), (3, 4, 1)$

Round 5

1. 52
2. $\frac{5}{13}$
3. $2 - \sqrt{3}$

Round 6

1. 0, 90
2. $\frac{5}{3}$ or 5:3
3. $\frac{4029\pi}{2}$

Team

1. 10
2. −1
3. $(4, 22, -9)$
4. $\frac{41}{504}$
5. 3200
6. 5

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

Solutions: State Meet 2013

Round 1 Arithmetic and Number Theory:

1. $\dfrac{\$6}{4.5\text{ lb}} = \dfrac{\$4}{3}$ per pound. $\dfrac{\$7.5}{3\text{ lb}} = \$2.50$ per pound. Finally, $\dfrac{5/2}{4/3} = \dfrac{15}{8} = 1.875$. Thus the price per pound increased by $\boxed{87.5\%}$.

2. We have $p^2 - 12p + 35 = (p-5)(p-7)$. Here $p - 7$ must equal 1, making $p = 8$ and so $p^2 - 12p + 35 = 3$. Or $p - 5 = -1$, making $p = 4$, thereby making $p - 7 = -3$, giving the product of 3. So $p = \boxed{4 \text{ or } 8}$.

3. $\dfrac{10A + A}{75} = \dfrac{B}{10} + \dfrac{B}{100} \Rightarrow \dfrac{11A}{75} = \dfrac{11B}{100} \Rightarrow \dfrac{4A}{3} = B$.
If $A = 3$, then $B = 4$ and if $A = 6$, then $B = 8$. Thus, $(A, B) = \boxed{(3,4),\ (6,8)}$.

Round 2 Algebra I:

1. The total cost is nk. If he sells x items at a price of $k\left(1+\dfrac{p}{100}\right)$ we have $k\left(1+\dfrac{p}{100}\right)x = nk \Rightarrow$ $\left(1+\dfrac{p}{100}\right)x = n \Rightarrow x = \dfrac{100n}{100+p}$. If x is not an integer, then we must round the value of this expression down and add 1 to obtain the minimum. Thus, $Q = \boxed{\dfrac{100n}{100+p}}$.

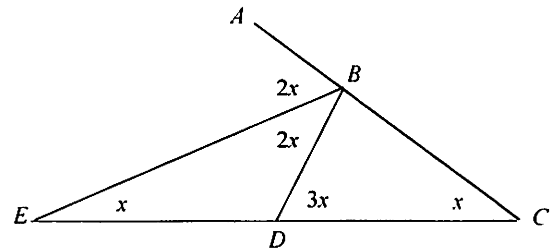
2. The equation is quadratic in $\sqrt[4]{x}$ so we have $\left(\sqrt[4]{x}\right)^2 + 28\sqrt[4]{x} - 2013 = 0 \Rightarrow$ $\left(\sqrt[4]{x} - 33\right)\left(\sqrt[4]{x} + 61\right) = 0 \Rightarrow \sqrt[4]{x} = 33 \Rightarrow x = 33^4 = (3\cdot 11)^4$. Thus, $\boxed{(a,b,c) = (3,11,4)}$.

3. From $a^2 - b^2 = 2013^2$ we factor and obtain $(a+b)(a-b) = 1^2 \cdot 3^2 \cdot 11^2 \cdot 61^2$. There are $3\cdot 3\cdot 3 = 27$ factors, making 13 pairs of unequal factors and one pair of equal factors, (2013, 2013). Setting $a+b$ equal to the larger factor and $a-b$ equal to the smaller factor, one can find all ordered pairs of solutions (a,b) as long as the factors have the same parity. In this problem all factors are odd so all factor pairs have the same parity so all factor pairs give solutions. Thus, there are $\boxed{13}$ solutions.

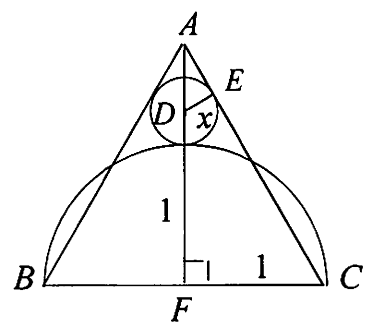
Round 3 Geometry:

1. Since $\frac{a(\Delta CDE)}{a(\Delta CAB)} = \frac{1}{2}$, then $\frac{CD}{AC} = \frac{1}{\sqrt{2}} \Rightarrow \frac{CD}{CD+4} = \frac{1}{\sqrt{2}} \Rightarrow CD\sqrt{2} = CD + 4$. Solving, we obtain $CD = \frac{4}{\sqrt{2}-1} = \boxed{4(\sqrt{2}+1)}$.

2. Let $m\angle E = m\angle C = x$. By the exterior angle theorem, $m\angle ABE = 2x$, making $m\angle EBD = 2x$. Again by the exterior angle theorem, $m\angle BDC = 3x$. Since $\angle BDC$ is acute, $3x < 90$, making $x < 30$. Thus, the largest integral value for the measure of angle E is $\boxed{29}$.



3. The radius of the semicircle is 1. Let the radius of the small circle be x. ΔADE is a 30-60-90 so $AD = 2x$. $\overline{AF}$ is the altitude of ΔABC so $AD = \sqrt{3} = 1 + 3x$. Thus, $\boxed{x = \frac{\sqrt{3}-1}{3}}$



Round 4 Algebra II

1. $3 \cdot 4^x + 6y \cdot 2^x - 45y^2 = 3(2^{2x} + 2y \cdot 2^x - 15y^2) = \boxed{3(2^x - 3y)(2^x + 5y)}$

2. Under the transformation the points become $A = (-6, -5)$, $B = (-3, 1)$, $C = (0, -2)$, and $D = (4, 10)$. The largest zero will be the x-intercept of $\overline{CD}$. The slope of $\overline{CD}$ is $\frac{10-(-2)}{4-0} = 3$, making the equation $y = 3x - 2$. The x-intercept is $\boxed{\frac{2}{3}}$.

3. Subtracting gives $2x + z = 7 \Rightarrow z = 7 - 2x$. Let $x = t$ and $z = 7 - 2t$. Substituting into either of the original equations gives $y = t + 1$. From $x = t$ we obtain $t > 0$ and from $z = 7 - 2t$ we obtain $7 - 2t > 0 \Rightarrow t < 3.5$. Thus t can take on the values of 1, 2, and 3, giving 3 ordered triples of positive integers. Those triples are $\boxed{(1, 2, 5), (2, 3, 3), \text{and } (3, 4, 1)}$.

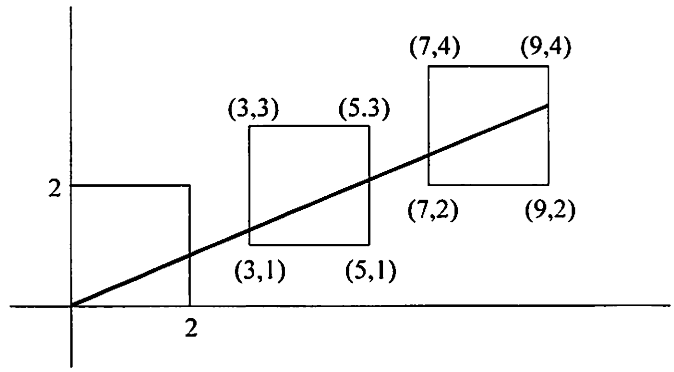
Round 5 **<u>Analytic Geometry:</u>**

1. The intercepts of $f(x) = mx + b$ are $(0,b)$ and $\left(-\frac{b}{m},0\right)$. Thus, $\frac{1}{2}\cdot b\cdot\frac{-b}{m} = 13 \rightarrow -\frac{b^2}{m} = 26$.

   From $y = 2f\left(\frac{x}{2}\right) = 2\left(\frac{mx}{2}+b\right) = mx + 2b$ we obtain intercepts $(0,2b)$ and $\left(-\frac{2b}{m},0\right)$, making the area equal to $\frac{1}{2}\left(-\frac{2b}{m}\right)(2b) = -\frac{2b^2}{m} = 2\cdot 26 = \boxed{52}$.

   Alternate Solution: From area of a triangle $\frac{1}{2}bh = 13$, so $\frac{1}{2}(2b)(2h) = 4\cdot 13 = 52$

2. The three regions consist of a triangle plus two trapezoids. If the equation of the line is $y = mx$, the height of the triangle is $2m$, making its area equal to $\frac{1}{2}\cdot 2\cdot 2m$. The height of the first trapezoid is 2, the short base is $3m-1$ and the long base is $5m-1$. Its area is $2\cdot\left(\frac{(3m-1)+(5m-1)}{2}\right)$.



   We find the area of the second trapezoid similarly. The sum of the areas is
   $\frac{1}{2}\cdot 2\cdot 2m + 2\cdot\left(\frac{3m-1+5m-1}{2}\right) + 2\cdot\left(\frac{7m-2+9m-2}{2}\right) = \frac{1}{3}\cdot 12$.

   This simplifies to $26m = 10$, giving $\boxed{m = \frac{5}{13}}$.

3. $AB = 2\sqrt{5}$. The circles with centers at A and B and radius $2\sqrt{5}$ will intersect at C. Solving the system $x^2 + (y-5)^2 = 20$ and $(x-4)^2 + (y-3)^2 = 20$ gives $y = 2x$. Substituting into the first equation gives $x^2 + (2x-5)^2 = 20 \Rightarrow x^2 - 4x + 1 = 0$. Solving gives
   $x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. We choose $\boxed{2-\sqrt{3}}$.

Round 6 **Trigonometry and Complex Numbers:**

1. $1 + \cos 2x = 2\sin(90° - x) \rightarrow 2\cos^2 x = 2\cos x \rightarrow \cos x(\cos x - 1) = 0$ $\boxed{0°, 90°}$

2. Let $x =$ the other side. Using the Law of Cosines $7^2 = x^2 + 64 - 2 \cdot 8 \cdot x \cdot \cos 60$, giving $x^2 - 8x + 15 = 0 \Rightarrow x = 3$ or 5. Since the triangles have the same height, the ratio of their areas is the ratio of their bases so the answer is $\boxed{\frac{5}{3} \text{ or } 5:3}$.

3. Let $x = 2y$ giving $\sin 3y - \sin y = 0 \rightarrow 3\sin y - 4\sin^3 y - \sin y = 0 \Rightarrow (2\sin y)(1 - 2\sin^2 y) = 0$. Thus, $\sin y = 0 \Rightarrow y = 0 + \pi k \Rightarrow \frac{x}{2} = 0 + \pi k \Rightarrow x = 0 + 2\pi k \Rightarrow$ the largest solution in the interval is 2013π. Or $\sin y = \pm\frac{\sqrt{2}}{2} \Rightarrow y = \frac{\pi}{4} + \frac{\pi}{2}k \Rightarrow x = \frac{\pi}{2} + \pi k$. From this set of solutions the largest in $[2013\pi, 2015\pi)$ is $2015\pi - \frac{\pi}{2} = \boxed{\frac{4029\pi}{2}}$.

<u>Alternate solution</u>: Since $\sin\frac{A+B}{2} - \sin\frac{A-B}{2} = 2\cos\frac{A}{2}\sin\frac{B}{2}$, then if we let $\frac{A+B}{2} = 3x$ and $\frac{A-B}{2} = \frac{x}{2}$, we have $A = 2x$ and $B = x$, giving $2\cos x \sin\frac{x}{2} = 0$. Thus, $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + \pi k$ or $\sin\frac{x}{2} = 0 \Rightarrow \frac{x}{2} = 0 + \pi k \Rightarrow x = 0 + 2\pi k$ and the solution is as above.

Team:

1. From the number ABC we have $Ax^2 + Bx + C = 0$, where $B^2 - 4AC = 0$. We note that B must be even. If $B = 0$, then $AC = 0$. Since $A \neq 0$, then $C = 0$ and A can take on all nine values from 1 to 9, giving 9 solutions, but we reject those. If $B = 2$, then $4 - 4AC = 0 \Rightarrow AC = 1 \Rightarrow A = 1, C = 1$, giving the number 121. If $B = 4$, then $16 - 4AC = 0 \Rightarrow AC = 4 \Rightarrow (A,C) = (1,4),(2,2),(4,1)$, giving the numbers 144, 242, and 441. If $B = 6$, then $36 - 4AC = 0 \Rightarrow AC = 9 \Rightarrow (A,C) = (1,9)$, $(3,3)$, or $(9,1)$, giving the numbers 169, 363, and 961. Finally, if $B = 8$, then $AC = 16$, giving $(A,C) = (4,4)$, $(2,8)$, or $(8,2)$, resulting in 484, 288, or 882. Thus, there are $1 + 3 + 3 + 3 = \boxed{10}$ three-digit numbers without a zero in the ten's place whose digits give a quadratic with a double root.

2. $f(f(f(2))) = f(f(2m+b)) = f(2m^2+mb+b) = 2m^3+m^2b+mb+b$. We obtain $2m^3+m^2b+mb+b = 2m+b \Rightarrow$
$2m^3+m^2b+mb = 2m \rightarrow 2m^2+mb+b = 2 \Rightarrow$
$2m^2+mb+(b-2) = 0 \Rightarrow m = \dfrac{-b \pm \sqrt{b^2-4\cdot 2(b-2)}}{4} = \dfrac{-b \pm \sqrt{b^2-8b+16}}{4} =$
$\dfrac{-b \pm |b-4|}{4}$. Since $b > 4$, $m = \dfrac{-b \pm (b-4)}{4} = \dfrac{-4}{4}$ or $\dfrac{-2b+4}{4}$. For $b > 4$,
$y = -\dfrac{b}{2}+1$ lies below $y = -1$, so the maximum value of m is $\boxed{-1}$.

3. Note that ΔABC is a right triangle and since $AC = 20$ and $AB = 12$, then $BC = 16$. Let $m\angle ACB = \theta$, then $\cos\theta = \dfrac{4}{5}$. The measure of $\angle DCA = 60-\theta$. Using the Law of Cosines on ΔDCA gives $AD^2 = 12^2+20^2-2\cdot 12\cdot 20\cos(60-\theta) = 544-480(\cos 60\cos\theta+\sin 60\sin\theta)$. Since $\cos\theta = \dfrac{4}{5}$ and $\sin\theta = \dfrac{3}{5}$, we have $AD^2 = 544-480\left(\dfrac{1}{2}\cdot\dfrac{4}{5}+\dfrac{\sqrt{3}}{2}\cdot\dfrac{3}{5}\right) =$
$352-144\sqrt{3} = 16\left(22-9\sqrt{3}\right)$. Thus, $AD = 4\sqrt{22-9\sqrt{3}}$, making $\boxed{(a,b,c) = (4,22,-9)}$.

4. The only numbers that can be flanked by its factors are 6, 8, and 10. An example of a correct sequence is 2, 6, 3, 4, 5, 7, 8, 9, 10. There are 3 numbers that can be flanked by their factors, the two factors can switch sides, there are 7 places for the flanked number to be, and there are $6!$ possible arrangements of the remaining 6 numbers, making a total of $3\cdot 2\cdot 7\cdot 6! = 6\cdot 7!$ possible sequences satisfying the conditions. Unfortunately, this over counts sequences involving the following six quintuples: 36284, 48263, 362(10)5, 5(10)263, 482(10)5, and 5(10)284. The middle 2 can be placed in any one of 5 places, places 3 through 7 in the
sequence, and the other four numbers can be arranged in $4!$ ways. So we subtract $6\cdot 5\cdot 4! = 6!$ from $6\cdot 7!$ to obtain the total number of arrangements in which at least one number is flanked by its factors. The probability that an arrangement will satisfy the conditions is $\dfrac{6\cdot 7!-6!}{9!} = \boxed{\dfrac{41}{504}}$.

5. Consider $(5x \cdot 4y) \cdot (120 - 5x - 4y)$. It is the product of three numbers with a constant sum of 120 so by the AM-GM we know that $\frac{(120 - 5x - 4y) + 5x + 4y}{3} \geq \sqrt[3]{5x \cdot 4y \cdot (120 - 5x - 4y} \Rightarrow$ $40 \geq \sqrt[3]{5x \cdot 4y \cdot (120 - 5x - 4y)}$ so the largest value of $5x \cdot 4y \cdot (120 - 5x - 4y)$is 40^3. Thus the largest value of $xy(120 - 5x - 4y)$ is $\frac{40^3}{20} = \boxed{3200}$.

Alternately, we could have solved for x and y since the largest value occurs when the terms are equal. This gives the equations $5x = 120 - 5x - 4y$and $4y = 120 - 5x - 4y$. Solving we obtain $x = 8$ and $y = 10$ giving a product of$8 \cdot 10 \cdot (120 - 40 - 40) = 80 \cdot 40 = 3200$.

Alternate 2: Using calculus, let $f = xy(120 - 5x - 4y)$, set $\frac{df}{dx} = 2y(60 - 5x - 2y) = 0$ and $\frac{df}{dy} = x(120 - 5x - 8y)$. Now $5x + 2y = 60$ and $5x + 8y = 120$. Subtracting gives $6y = 60$. Therefore $y = 10$ and $x = 8$.

6. By the parallelogram law, $2\left(6^2 + 6^2\right) = 2^2 + DB^2 \Rightarrow DB = 2\sqrt{35}$, making $DT = \sqrt{35}$. Or one could drop the altitude from B meeting $\overline{AC}$ at T. Then $AT = 1$and $6^2 - 1^2 = BT^2 \to BT = \sqrt{35}$. Since $AC = 2$then $CT = 1$. Tangents to a circle from an external point are equal, so $CE = 1$, making $BE = 5$. Let the radius of the circle be r, making $BO = \sqrt{35} - r$. Then

$$r^2 + 5^2 = \left(\sqrt{35} - r\right)^2 \Rightarrow 2r\sqrt{35} = 10 \Rightarrow r = \frac{5}{\sqrt{35}}.$$

Then the product of the radii is $\sqrt{35} \cdot \frac{5}{\sqrt{35}} = \boxed{5}$.