LEAGUES

MAML

STATE INVITATIONAL MATH LEAGUE COMPETITION

April 3, 2009

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2009

Round 1 Arithmetic and Number Theory

1. __________________

2. __________________

3. __________________

1. Let p and q be primes with $p < q < 20$. If the sum $p + q$ is prime, determine the number of possible ordered pairs (p, q).

2. Find the <u>smallest</u> positive integer n such that n^2 is divisible by 243 and ends in 25.

3. Numbers A and B are <u>not</u> relatively prime. If A has 9 positive factors and B has 6 positive factors, determine the greatest number of positive factors that the product $A \cdot B$ could have.

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Round 2 **Algebra 1**

1. ____________________

2. ____________________

3. ____________________

1. Eighty students came out for tennis. They ~~are~~ WERE to pair up and play 4 singles matches per week with the same partner. Each match lasts 45 minutes each, making for 120 hours of court time. However, ten students decided that they ~~wanted to~~ WOULD INSTEAD play all their matches with friends who were not in the group of 80. If they meet their time requirement, what is the increase in court time per week?

2. Determine the absolute value of the product ab given the system:

$$a + \frac{3}{b} = 1$$

$$b - \frac{1}{a} = 2$$

3. Determine the largest value of n for which $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) > \frac{1005}{2009}$.

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Round 3 – Geometry

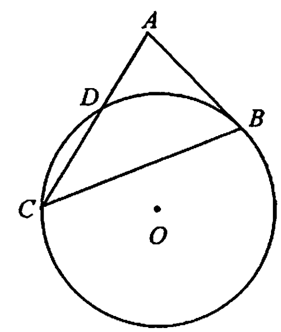
1. ______________________

2. ______________________

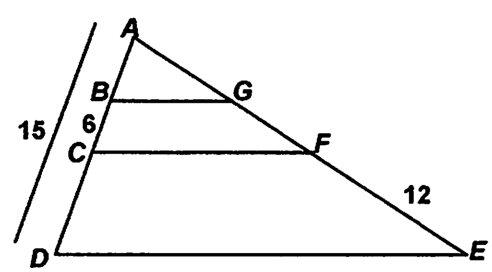
3. ______________________

1. For distinct points A, B, C, and D, B is the midpoint of $\overline{AC}$. If D lies on $\overline{BC}$ such that $AD = 3x + 1$, $DC = x + 2$, and $AC > k$. Determine the largest possible value of k.

2. $\overline{AB}$ is tangent to circle O, $m\angle C = 30°$, and $m\angle A = 84°$. Find $m\overset{\frown}{CD}$.



3. $\overline{BG}$ and $\overline{CF}$ are parallel to $\overline{DE}$, $BC = 6$, $AD = 15$, $FE = 12$, $AB > 0$, and $m\angle D > m\angle E$.
For $GF \leq 2009$, determine the number of integral values that GF can take on.



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Round 4 – Algebra 2

1. ________________

2. ________________

3. ________________

1. Given the arithmetic sequence $\{a_i\}$ with $a_1 = 4$ and $a_8 = 9$, determine the number of values of n, $1 \le n \le 2009$, such that a_n is an integer.

2. If the ratio of the solutions in x to $\log_b x - 4\log_x b = 3$ is $4\sqrt{2}:1$, find the larger value of b.

3. The following numbers are written randomly in a row: 1, 2, 3, 5, 8, 13, 21, 34 and 55. Determine the probability that the sum of the first and the last terms equals the middle term.

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Round 5 – Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. The parametric equations $x = t^2 - 1$ and $y = 4t - 2$ represent a parabola. Determine the y-intercepts.

2. Given points $A(0, 4)$, $B(2, 0)$, and $P(k, 0)$, determine the value of k such $PA = 2AB$.

3. The area of the triangular region bounded by $y = x$, $y = -2x + 8$, and the x-axis is A. Find k so that the area bounded by $y = kx$, $y = -2x + 8$, and the x-axis is $2A$.

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Round 6 – Trig and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

1. If $\text{Sin}^{-1}\left(\sin\frac{19\pi}{6}\right)=\frac{1}{x}$, determine the value(s) of x.

2. Determine the number of complex numbers z such that $z^2 = \bar{z}$ where $\bar{z}$ is the conjugate of z.

3. Let x be in radian measure. Determine the number of solutions in the interval $0 \le x < 4\pi$ to the equation $\sin x \cos^2 x = \frac{1}{4}\sin x$.

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Team Round

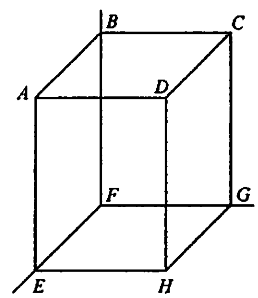
1. ______________ **4.** ______________

2. ______________ **5.** ______________

3. ___(_____,_____)___ **6.** ______________

1. The integers 1, 2, 3, 4, 5, and 6 are written down randomly in a row. Determine the probability that in the row there is exactly one sequence of exactly three consecutive integers in increasing order.

2. Given $D(6, 8, 24)$, form a rectangular box by drawing lines parallel to the axes as indicated in the diagram. The equation of the smallest sphere which circumscribes DFDG may be expressed in the form $(x-h)^2+(y-j)^2+(z-k)^2=r^2$. Determine the ordered quadruple (h,j,k,r).



3. Find the maximum value of $\sqrt{(x-4)^2+(x^2-1)^2}-\sqrt{x^2+(x^2-2)^2}$.

4. Consider the triangular numbers $T_n=1+2+\cdots+n$ and $T_m=1+2+\cdots+m$ with $n>m$. Find all ordered pairs (n,m) such that $T_n-T_m=2009$.

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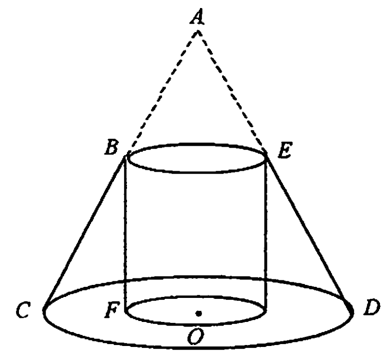
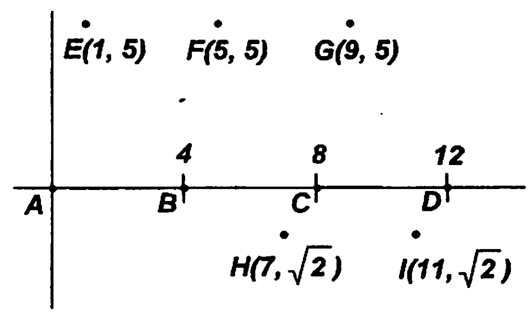
STATE PLAYOFFS – 2009

Team Round - Page 2

5. Let Ω be a cone whose radius is 6 and whose height is 8. The top is lopped off the cone, turning it into the dreaded frustum. Then a cylindrical hole is drilled out right through the center, turning it into that which cannot be named, namely Θ. Find the height, BF, of the hole if the lateral surface area of Θ equals the original lateral surface area of Ω.



6. How many non-degenerate triangles can be formed whose vertices are three of the nine points pictured?



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Answer Sheet

Round 1

1. 4
2. 135
3. 36

Round 2

1. 15
2. $\sqrt{3}$
3. 2008

Round 3

1. 5
2. 72
3. 2001

Round 4

1. 287
2. $\sqrt{2}$
3. $\frac{1}{36}$

Round 5

1. 2, −6
2. −8, 8
3. 4

Round 6

1. $-\frac{6}{\pi}$
2. 4
3. 12

Team

1. $\frac{31}{360}$
2. $(3,4,12,13)$
3. $\sqrt{17}$
4. $(2009, 2008)$, $(1005,1003)$, $(290,283)$, $(150,136)$, $(69,28)$, $(65,16)$
5. $\frac{40}{13}$
6. 79

Solutions: State Meet 2009

Round 1 Arithmetic and Number Theory:

1. If the sum of two primes is a prime number, one of the primes must be 2. The possibilities are $2+3=5$, $2+5=7$, $2+11=13$, and $2+17=19$. Ans: $\boxed{4}$

2. Since $\left(3^2\right)^2 = 81 < 243$, n must have 3^3 as a factor along with 5. Then $n = 5\cdot 3^3 = \boxed{135}$.

3. A could equal $p_1^{\,8}$ or $p_1^{\,2}p_2^{\,2}$. B could equal $p_1^{\,5}, p_1^{\,2}p_2, p_1p_2^{\,2}, p_1^{\,2}p_3$, or $p_1p_3^{\,2}$. We can ignore $p_2^{\,2}p_3$ and $p_2p_3^{\,2}$ since they will generate the same outcomes as $p_1^{\,2}p_3$ and $p_1p_3^{\,2}$. Shown are the possible products and, in parentheses, the number of factors.

$A\cdot B = p_1^{\,13}$ (14), $p_1^{\,10}p_2^{\,2}$ (22), $p_1^{\,9}p_2^{\,2}$ (30), $p_1^{\,7}p_2^{\,2}$ (24), $p_1^{\,4}p_2^{\,3}$ (20), $p_1^{\,3}p_2^{\,4}$ (20), $p_1^{\,10}p_3$ (22), $p_1^{\,4}p_2^{\,2}p_3$ (30), $p_1^{\,9}p_3^{\,2}$ (30), or $p_1^{\,3}p_2^{\,2}p_3^{\,2}$ (36). Answer: $\boxed{36}$.

Round 2 Algebra I:

1. If $2n$ students played with someone else, then total time is $\dfrac{80-2n}{2}\cdot 3 + 2n\cdot 3 = 120 - 3n + 6n = 120 + 3n$. Since $n = 5$, the increase in court time is $\boxed{15}$ hours.

2. Multiply the equations together obtaining $ab + 3 - 1 - \dfrac{3}{ab} = 2 \to ab = \dfrac{3}{ab} \to (ab)^2 = 3 \to |ab| = \boxed{\sqrt{3}}$.

3. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\cdots\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right) =$
$\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n}\right) = \frac{1}{n}\cdot\frac{n+1}{2}$. Thus, we want to find the largest n such that $\dfrac{n+1}{2n} > \dfrac{1005}{2009}$. We obtain $2009n + 2009 > 2010n \to n < 2009$. Thus $\boxed{n = 2008}$.

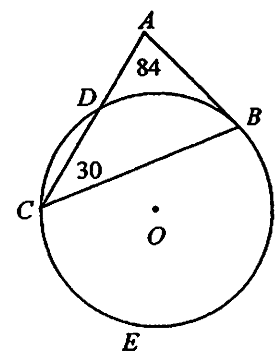
Round 3 Geometry:

1. The length of $\overline{AC}$ is $4x+3$. Since $AD > DC$, then $3x+1 > x+2 \to 2x > 1 \to x > \frac{1}{2}$. The greatest lower bound for AC is $4(1/2)+3 = \boxed{5}$.

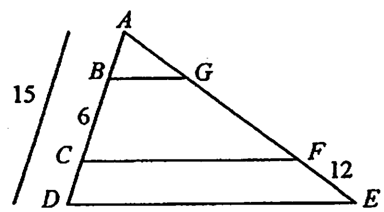
Round 3 Geometry – continued

2. $\frac{m\overset{\frown}{CEB} - m\overset{\frown}{DB}}{2} = 84 \rightarrow m\overset{\frown}{CEB} = 168 + m\overset{\frown}{DB}$. Since $m\overset{\frown}{DB} = 2m\angle C$ then $m\overset{\frown}{DB} = 60$, making $m\overset{\frown}{CEB} = 228$.

   Then $m\overset{\frown}{CD} = 360 - 228 - 60 \rightarrow \boxed{m\overset{\frown}{CD} = 72}$.



3. If $GF = 8$, then $CD = 9$ making $AB = 0$ but $AB > 0$. So $GF > 8$ but since $m\angle D > m\angle E$ then there is no upper limit on GF. Thus, GF can take on all integer values from 9 to 2009 inclusive. The number is $2009 - 9 + 1 = \boxed{2001}$. Note: CD need not be an integer; otherwise GF would be constrained.



Round 4 Algebra II:

1. $a_8 = a_1 + 7d \rightarrow 9 = 4 + 7d \rightarrow d = \frac{5}{7}$. Hence $a_n = 4 + (n-1)\frac{5}{7}$. By inspection a_n is an integer whenever $n - 1 = 6, 13, 20, \ldots, 2008$. There are $\frac{2008-6}{7} + 1 = \boxed{287}$ terms.

2. $(\log_b x)^2 - 3\log_b x - 4 = 0 \rightarrow (\log_b x - 4)(\log_b x + 1) = 0 \rightarrow x = b^4$ or b^{-1}. The ratio $b^4 : b^{-1}$ gives b^5 so $b^5 = 4\sqrt{2} = 2^{5/2} \rightarrow b = \sqrt{2}$. The ratio $b^{-1} : b^4$ gives $b^{-5} = 2^{5/2} \rightarrow b = 2^{-1/2}$. Thus, the larger value of $\boxed{b = \sqrt{2}}$.

3. Since these are the Fibonacci numbers, each selection of three consecutive numbers would satisfy the requirement if the first two were the first and last in the sequence of nine numbers and the third was the middle term. In the nine numbers there are 7 such selections. Since the smaller two terms can be interchanged, and the remaining six terms can be permuted, the probability is $\frac{7 \cdot 2 \cdot 6!}{9!} = \frac{2}{9 \cdot 8} = \boxed{\frac{1}{36}}$.

Solutions: State Meet 2009

Round 5 Analytic Geometry:

1. $\begin{cases} x = t^2 - 1 \\ y = 4t - 2 \end{cases} \rightarrow x = \left(\frac{y+2}{4}\right)^2 - 1$ To find the y-intercepts, let $x = 0$ and solve for y.

   $0 = \left(\frac{y+2}{4}\right)^2 - 1 \rightarrow y = -2 \pm 4 \rightarrow \boxed{2, -6}$

2. Let $C = (x, 0)$ be the point on the x-axis. Then $PA = 2PB$ gives $\sqrt{(x-0)^2 + (4-0)^2} = 2\sqrt{(0-2)^2 + (4-0)^2} \rightarrow x^2 + 16 = 80$ giving $x = \pm 8$.
   Thus, $\boxed{k = -8 \text{ or } 8}$.

3. Solving $x = -2x + 8$ we obtain $\left(\frac{8}{3}, \frac{8}{3}\right)$ for the intersection point of the two lines. The height of the triangle is 8/3 so $A = \frac{1}{2} \cdot \frac{8}{3} \cdot 4 = \frac{16}{3}$. The x-coordinate of the intersection point of $y = kx$ and $y = -2x + 8$ is found by solving $kx = -2x + 8 \rightarrow x = \frac{8}{k+2}$. Then $y = \frac{-16}{k+2} + 8 = \frac{8k}{k+2}$. That is the height of the second triangle and it must be twice the height of the first triangle since they have the same base. Thus $\frac{8k}{k+2} = 2 \cdot \frac{8}{3} \rightarrow \frac{k}{k+2} = \frac{2}{3}$ giving $\boxed{k = 4}$.

Round 6 Trigonometry and Complex Numbers:

1. $\text{Sin}^{-1}\left(\sin\frac{19\pi}{6}\right) = \text{Sin}^{-1}\left(\sin\frac{7\pi}{6}\right) = \text{Sin}^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} = \frac{1}{x}$. Thus, $x = \boxed{-\frac{6}{\pi}}$

2. $(a + bi)^2 = a - bi \rightarrow (a^2 - b^2) + (2ab)i = a - bi \rightarrow a^2 - b^2 = a$ and $2ab = -b$. Choosing the second equation we have $b(2a + 1) = 0 \rightarrow b = 0$ or $a = -\frac{1}{2}$. If $b = 0$ then substituting into the first equation gives $a^2 = a \rightarrow a = 0, 1$. Answers are $0 + 0i$ and $1 + 0i$. If $a = -\frac{1}{2}$ substituting into the first gives $\frac{1}{4} - b^2 = -\frac{1}{2} \rightarrow b = \pm\frac{\sqrt{3}}{2}$. Answers are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus $\boxed{4}$ complex numbers satisfy the equation.

3. $\sin x \cos^2 x = \frac{1}{4}\sin x \rightarrow \sin x = 0$ or $\cos^2 x = \frac{1}{4}$. From the former we obtain solutions of $0, \pi, 2\pi,$ and 3π. From the latter we know there are 4 solutions in each period so there are a total of $\boxed{12}$ solutions.

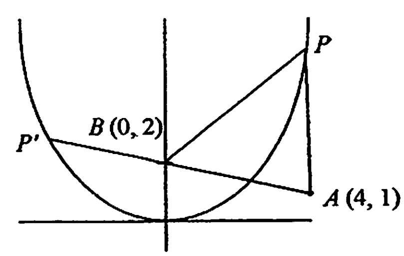
Solutions: State Meet 2009

Team Round:

1. Consider 123XXX. There are 3! ways to place 4, 5, and 6 in the remaining spaces, but 4 can't go immediately right of 3, so there are only $3! - 2 = 4$ ways to place 4, 5, and 6 in the remaining spaces. The same analysis holds for X123XX and XX123X. In the case of XXX123, there are $3! - 1 = 5$ ways to place 4, 5, and 6 in the spaces since 456 is not allowed. The total for 123 is 17. For 234 we have 4 ways for each of 234XXX and XXX234 since a 5 can't directly follow a 4 and 1 can't be immediately left of 2. For X234XX there are 3 ways since we can't have 123456, 123465 or 623451. Similarly, for XX234X there are 3 ways since we can't have 512346, 612345, or 162345. The total is 14. For 345, we have 4 ways for 345XXX since a 6 can't following the 5, 3 ways for each of X345XX and XX345X, and 4 ways for XXX345 since a 2 can't be immediately to the left of 3. The total is 14. For 456 there are 5 ways given 456XXX since 123 can't follow 456, and 4 ways for each of X456XX and XX456X since 3 can't directly precede 4, and 4 ways for XXX456 since 3 can't directly precede the 4. The total is 17. The final total is $17 + 14 + 14 + 17 = 62$, so the probability is $\frac{62}{6!} = \boxed{\frac{31}{360}}$.

2. Since DFG is a right angle, $\overline{DF}$ is the hypotenuse and the center of the sphere is the midpoint of $\overline{DF}$, namely $P(3, 4, 12)$. Since $PF = 13$, the equation of the sphere is $(x-3)^2 + (y-4)^2 + (z-12)^2 = 169 \rightarrow \boxed{(3, 4, 12, 13)}$

3. If we let $y = x^2$ the problem becomes: find the maximum value of $\sqrt{(x-4)^2 + (y-1)^2} - \sqrt{x^2 + (y-2)^2}$. This is the difference in the distances from point $P(x, x^2)$ on parabola $y = x^2$ and points $A(4,1)$ and $B(0,2)$. That is, find the maximum of $PA - PB$. Using ΔABP we have $PA < PB + AB$ giving $PA - PB < \sqrt{17}$. The question is, can the difference equal $\sqrt{17}$? If P lies on the intersection of AB and the parabola, which it does at P', then we have equality, namely the maximum of $PA - PB$ equals $\boxed{\sqrt{17}}$.



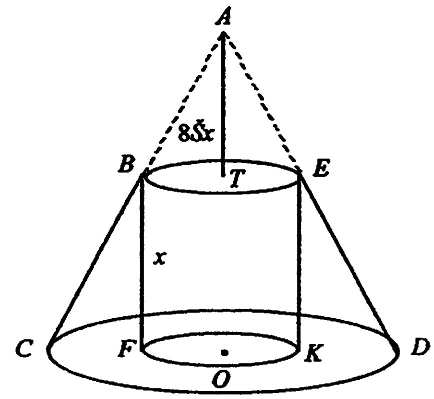
4. $\frac{1}{2}n(n+1) - \frac{1}{2}m(m+1) = 2009 \rightarrow n^2 - m^2 + n - m = 4018 \rightarrow$ $(n-m)(n+m+1) = 4018 = 2 \cdot 7^2 \cdot 41$. Setting $n + m + 1$ equal to the larger factor of 4018 and $n - m$ equal to the smaller we have the following table:

$n+m+1$	4018	2009	574	287	98	82
$n-m$	1	2	7	14	41	49
$2n =$	4018	2010	580	300	138	130

Solutions (n, m) are $\boxed{(2009, 2008), (1005, 1003), (290, 283), (150, 136), (69, 28), (65, 16)}$.

Team Round - continued

5. Since the cone and the frustum with the hole share the lateral area of the frustum we ignore that area. This means that the lateral area of cone *ABE* must equal the lateral area of cylinder *BEKF*. Let $BF = x$, let the radius of the cylinder and *ABE* equal *r*, the height of $ABE = 8 - x$ and because a



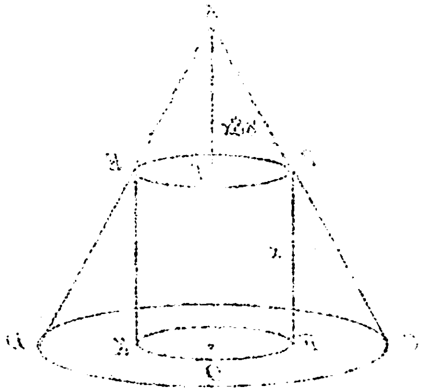
3-4-5 triangle is involved, $AE = \frac{5}{4}(8 - x)$. Thus, $2\pi rx = \pi r\frac{5}{4}(8 - x)$ giving $8x = 40 - 5x$ so $\boxed{x = \frac{40}{13}}$.

6. For each point on the *x*-axis there are ${}_5C_2 = 10$ pairs of points off the *x*-axis which will form a triangle. This gives 40 triples of points. There are ${}_4C_2 = 6$ pairs of points on the *x*-axis which can be combined with 1 of 5 points not on the *x*-axis giving 30 triples. Finally, the points not on the *x*-axis may be chosen in ${}_5C_3 = 10$ ways. But we have to reject the triple *EFG* since the points are collinear. There are $40 + 30 + 10 - 1 = \boxed{79}$ possible triangles.

Solutions: State Meet 2009

Team Round - continued

5. Since the cone and the frustum with the hole share the lateral area of the frustum we ignore that area. This means that the lateral area of cone ABE must equal the lateral area of cylinder BEXF. Let $BF = x$, let the radius of the cylinder and BG equal r, the height of ABE = $8 - x$ and because a



3-4-5 triangle is involved, $AB = \frac{5}{4}(8 - x)$. Thus, $2\pi rx = \pi r \cdot \frac{5}{4}(8 - x)$ giving $8x = 40 - 5x$ so $\boxed{x = \frac{40}{13}}$.

6. For each point on the x-axis there are ${}_5C_2 = 10$ pairs of points off the x-axis which will form a triangle. This gives 40 triples of points. There are ${}_4C_2 = 6$ pairs of points on the x-axis which can be combined with 1 of 5 points not on the x-axis giving 30 triples. Finally, the points not on the x-axis may be chosen in ${}_5C_3 = 10$ ways. But we have to reject the triple EFG since the points are collinear. There are $40 + 30 + 10 - 1 = \boxed{79}$ possible triangles.