MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2007

Round 1 **Arithmetic and Number Theory**

1. ____________________

2. ____________________

3. ____________________

1. When written in simplest form, how many fractions in the following set have a denominator less than 10?
$\left\{\frac{1}{10}, \frac{3/2}{10}, \frac{2}{10}, \frac{5/2}{10}, \frac{3}{10}, \frac{7/2}{10}, \ldots, \frac{19/2}{10}\right\}$

2. Determine the number of positive integer bases in which one can represent 150_{10} using only two digits, not necessarily distinct. (Note: expressions for digits greater than 9 are considered individual digits.)

3. Let 2005^3 be written as the product of two positive integers whose positive difference N is as small as possible. Determine N.

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Round 2 **Algebra 1**

1. ______________________

2. ______________________

3. ______________________

1. Solve: $x-\left(1-2\left(1-3\left(1-(4-x)\right)\right)\right)=-1$

2. Simplify: $\dfrac{\sqrt{168}+\sqrt{672}}{\sqrt{42}+\sqrt{378}}$.

3. The posts of a fence enclosing a rectangular field are uniformly spaced. If the length of each long side of the rectangle is tripled, keeping the same spacing, thereby increasing the number of fence posts by 72, determine the total number of fence posts used in the two long sides combined of the original fence.

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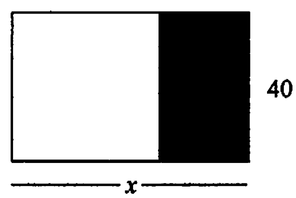
Round 3 – Geometry

1. ______________________

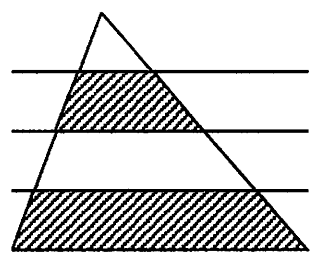
2. ______________________

3. ______________________

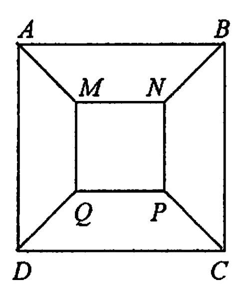
1. A rectangle 40 cm by 150 cm is painted white on the front face and black on the back face. When the rectangle is folded as shown, the ratio of the area of white portion to the area of the black portion is 3 to 1. Find the number of cm in the length x of the folded rectangle.



2. A triangular region is divided into four parts as shown. The three horizontal lines and base are parallel. The heights of the regions are equal. Determine the ratio of the sum of the areas of the shaded regions to the area of the whole region.



3. Square $ABCD$ is divided up into four congruent isosceles trapezoids and square $MNPQ$. If $AB = 14$ inches and the sum of the areas of $MNPQ$ and $ABNM$ is 97 square inches, determine the number of inches in MN.



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STATE PLAYOFFS – 2007

Round 4 – Algebra 2

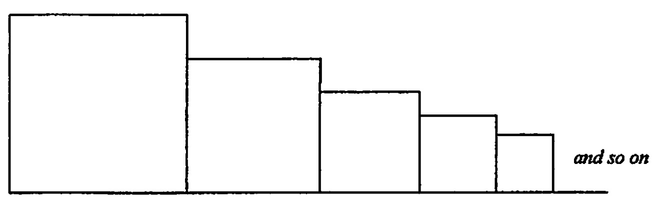
1. ________________________

2. ________________________

3. ________________________

1. Given the sequence of real numbers: 2, $\log_a b$, $\log_b a$, find the numerical value of $\log_a b$ if the sequence is geometric.

2. The length of the side of the left-most square is 16 units and the length of each side of each successive square is three-fourth's that of the length of each preceding square. Determine the number of units in the perimeter of the entire figure.



3. Find the number of real solutions to $\dfrac{1}{\sqrt{1+\sqrt{2x}}} = \sqrt{x}$.

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Round 5 – Analytic Geometry

1. ____________________

2. ____________________

3. ____________________

1. A circle with center in the first quadrant is tangent to the x- and y-axes as well as to the lines $y = x + 4$ and $y = x - 4$. Determine the exact number of units in the radius of the circle.

2. Determine the number of points at which the graph of $f(x) = \dfrac{2x^4 + 3x^3 - 2x^2 + 6x + 2}{x^4 + x^3 + x^2 + x + 1}$ intersects its horizontal asymptote.

3. Equilateral triangle ABC is inscribed in an ellipse whose equation is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ with A on the y-axis and $\overline{BC}$ parallel to the x-axis with C in the 4th quadrant. Given $D = (2, 0)$, determine the exact number of square units in the area of ΔADC.

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Round 6 – Trig and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. One solution to a quadratic equation $f(x)=0$ with real coefficients is $a-bi$, where a and b are integers, and $i=\sqrt{-1}$. If the coefficient of the x^2 term is 1, and the sum of the three coefficients is 18, determine all ordered pairs (a, b).

2. Determine the exact number of square units in the area of a triangle whose vertices in the complex plane are the origin and the first quadrant points given by $\sqrt{i}$ and $\sqrt[3]{i}$.

3. Find all values of k so that $x=\operatorname{Sin}^{-1}\left(\frac{1}{3}\right)$ is a solution to $\sin\frac{x}{2}+\cos\frac{x}{2}=k$.

   Note: $\operatorname{Sin}^{-1}\left(\frac{1}{3}\right)$ is the same as $\operatorname{Arcsin}\left(\frac{1}{3}\right)$

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Team Round

1. ____________________ 4. ____________________

2. ____________________ 5. ____________________

3. ____(______,______)____ 6. ____________________

1. A three-digit positive integer $1BC$ is *really swell* if its digits, in order from left to right, are the coefficients of a quadratic equation $1x^2 + Bx + C = 0$ that has two distinct real roots. Determine the number of *really swell* three-digit positive integers.

2. Given the set $T = \{1, 2, 3, \ldots, n\}$, the sums of all possible pairs of distinct numbers are computed and placed in a set S. Determine the least value of n so that the number of elements in S exceeds 2007.

3. Consider the following game: You start with 1 point and 28 chips. There are two ways in which you may increase your score. If you put in 2 chips, then 3 is added to your score. If you put 5 chips, then your score is doubled. What is the maximum score that you can obtain?

4. Let $S = \{1, 2, \ldots 3n + 1\}$ and $T = \{1, 2, \ldots, 3n\}$. Let x and y be the number of ways three distinct numbers can be chosen from the sets S and T respectively such that the sum of the three numbers is divisible by 3. The order in which the numbers are drawn is irrelevant. If $x - y = 330$, compute n.

5. Tangents to circle O are drawn from point P intersecting O at points A and B. Points A and B are vertices, not necessarily consecutive, of a regular n-sided polygon inscribed in circle O. If $m\angle APB = 33°$ and $n \leq 2007$, determine the number of values of n.

6. If a circle with center (0, 1) and radius 1 intersects a parabola whose equation is $y = ax^2$ in exactly one point, determine all possible values of a.

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Answer Sheet

Round 1

1. 7
2. 138
3. 110676

Round 2

1. 4
2. $\frac{3}{2}$
3. 38

Round 3

1. 120
2. $\frac{5}{8}$
3. 8

Round 4

1. $\sqrt[3]{2}$
2. 160
3. 1

Round 5

1. $2\sqrt{2}$
2. 2
3. $\frac{4\sqrt{3}}{5}$

Round 6

1. $(4,3),(4,-3),(-2,3),(-2,-3)$
2. $\frac{\sqrt{6}-\sqrt{2}}{8}$
3. $\frac{2\sqrt{3}}{3}$

Team

1. 55
2. 1006
3. 208
4. 15
5. 16
6. $a \le \frac{1}{2}, a \ne 0$

Solutions State Meet 2007

Round 1 Arithmetic and Number Theory:

1. All numerators that have a factor of 2 or 5 will yield a simpler denominator. Thus, if the numerators are 2, 4, 6, 8, 10, 5, 5/2, or 15/2, the denominator will be less than 10. Ans: 8.

2. Since $12^2 = 144$ and $13^2 = 169$, the least base must be 13. Indeed, $(11)7_{13} = 11 \cdot 13 + 7 = 150$. Since $(1)(0)_{150} = 150$ then the greatest base is 150. Hence there are $150 - 13 + 1 = 138$ bases that can represent 150 using two digits.

3. $2005^3 = 5^3 \cdot 401^3$. Hence, $N = 401^2 - 401 \cdot 125 = 401(401 - 125) = \boxed{110{,}676}$.

Round 2 Algebra I:

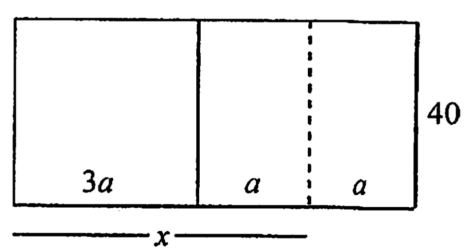
1. $x - (1 - 2(1 - 3(1 - (4 - x)))) = -1 \rightarrow x - (1 - 2(1 - 3(-3 + x))) = -1 \rightarrow$
$x - (1 - 2(10 - 3x)) = -1 \rightarrow x - (-19 + 6x) = -1 \rightarrow 19 - 5x = -1 \rightarrow -5x = -20$.
Thus, $x = 4$.

2. $$\frac{\sqrt{168} + \sqrt{672}}{\sqrt{42} + \sqrt{378}} = \frac{\sqrt{7}\sqrt{24} + \sqrt{7}\sqrt{96}}{\sqrt{7}\sqrt{6} + \sqrt{7}\sqrt{54}} = \frac{2\sqrt{6} + 4\sqrt{6}}{\sqrt{6} + 3\sqrt{6}} = \frac{3}{2}$$

3. Let the length of the original fence be x and the width be y. The lengthened fence will have dimensions $3x$ and y. The number of fence posts in the original fence will be $2(y - 1) + 2(x + 1)$ and the number in the lengthened fence will be $2(y - 1) + 2(3x + 1)$. We have $2(y - 1) + 2(x + 1) + 72 = 2(y - 1) + 2(3x + 1)$ since the new fence has 72 more posts. Thus, $x = 18$, making the number of posts in the two long sides in the original fence equal to 38.

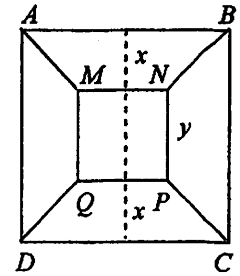
Round 3 Geometry:

1. Unfold the rectangle--the dotted line is the fold. Since the rectangles have a side of 40, then the areas are proportional to the lengths of the horizontal segments. Thus, the lengths are $3a$, a, and a, making $5a = 150$ so $4a = x = 120$.



2. The small shaded area is 3/4 of 1/4 the area of the whole triangle, making it 3/16 of the whole area. The bottom shaded region is $1 - 9/16 = 7/16$ the area of the whole triangle. The sum is 10/16, making the shaded areas equal to 5/8 the area of the whole triangle.

3. Let the height of each trapezoid be x and $NP = y$. From $2x + y = 14$ and $y^2 + \frac{1}{2} \cdot x(2y + 2x) = 97$, we replace y with $14 - 2x$ to obtain $x^2 - 14x + 33 = 0 \rightarrow$ $(x-3)(x-11) = 0 \rightarrow x = 3, 11$. Reject 11 since it makes y negative, so $x = 3$, making $y = \boxed{8}$.



Alternate Solution: Let s = the area of $MNPQ$ and t = the area of a trapezoid. Then $s + 4t = 196$ and $s + t = 97$ so $3t = 99 \rightarrow t = 33 \rightarrow s = 64$. Thus, $MN = 8$.

Round 4 Algebra II:

1. $\sqrt[3]{2}$ since $\frac{\log_a b}{2} = \frac{\log_b a}{\log_a b} = \frac{1}{(\log_a b)^2} \rightarrow (\log_a b)^3 = 2$

2. $2\left(16 + 16 \cdot \frac{3}{4} + 16 \cdot \left(\frac{3}{4}\right)^2 + \ldots\right) + 2 \cdot 16 = \frac{2 \cdot 16}{1 - \frac{3}{4}} + 32 = 160.$

3. $\frac{1}{\sqrt{1+\sqrt{2x}}} = \sqrt{x} \rightarrow 1 = x + x\sqrt{2x} \rightarrow x\sqrt{2x} = 1 - x \rightarrow 2x^3 = 1 - 2x + x^2$

$2x^3 = x^2 - 2x + 1 \rightarrow 2x^3 - x^2 + 2x - 1 = 0 \rightarrow x^2(2x-1) + (2x-1) = 0 \rightarrow$

$(x^2 + 1)(2x - 1) = 0$. Thus, $x = \frac{1}{2}$.

Round 5 Analytic Geometry:

1. Let the center of the circle be $O(r, r)$. Then the radius is r and the distance from O to $x - y + 4 = 0$ is also r. Thus, $\frac{|r - r + 4|}{\sqrt{1^2 + 1^2}} = r \rightarrow r = 2\sqrt{2}$.

2. $\frac{2x^4 + 3x^3 - 2x^2 + 6x + 2}{x^4 + x^3 + x^2 + x + 1} = 2 \rightarrow 2x^4 + 3x^3 - 2x^2 + 6x + 2 = 2x^4 + 2x^3 + 2x^2 + 2x + 2.$

Thus, $x^3 - 4x^2 + 4x = 0 \rightarrow x(x-2)^2 = 0$. Thus, $x = 0, 2$ and so there are $\boxed{2}$ points of intersection, one of which is a point of tangency.

3. Equation of line $\overline{AC}$: $y = -\sqrt{3}x + \sqrt{3} = \sqrt{3}(1-x)$. Substitute to find C:

$$\frac{x^2}{4} + \frac{3(1-2x+x^2)}{3} = 1 \rightarrow \frac{x^2}{4} + x^2 - 2x = 0 \rightarrow x(5x+8) = 0 \rightarrow x = 8/5 \text{ and}$$

$$y = -\frac{3\sqrt{3}}{5}. \text{ The area of } \Delta ADC = \frac{1}{2}\left(\begin{vmatrix} 2 & 0 \\ 0 & \sqrt{3} \end{vmatrix} + \begin{vmatrix} 0 & \sqrt{3} \\ \frac{8}{5} & -\frac{3\sqrt{3}}{5} \end{vmatrix} + \begin{vmatrix} \frac{8}{5} & -\frac{3\sqrt{3}}{5} \\ 2 & 0 \end{vmatrix}\right) =$$

$$\frac{1}{2}\left(2\sqrt{3} - \frac{8\sqrt{3}}{5} + \frac{6\sqrt{3}}{5}\right) = \frac{4\sqrt{3}}{5}.$$

Round 6 <u>Trigonometry and Complex Numbers:</u>

1. $x^2 - 2ax + (a^2 + b^2) = 0 \rightarrow a^2 - 2a + 1 + b^2 = 18$. Thus, $(a-1)^2 + b^2 = 18$. This is only true for $(\pm 3)^2 + (\pm 3)^2$. Hence $a - 1 = \pm 3$, giving $a = 4, -2$ and $b = \pm 3$. The ordered pairs are (4, 3), (4, –3), (–2, 3), and (–2, –3).

2. The points are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ but the important thing is that their distance from the origin is 1 and the first makes a 45° with the x-axis and the second makes a 30° so we have a triangle with two sides of 1 and an included angle of 15°. The area equals $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 15° = \frac{1}{2}\sqrt{\frac{2-\sqrt{3}}{4}} = \frac{1}{4}\sqrt{2-\sqrt{3}}$. Using $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$ we also have the equivalent answer $\frac{\sqrt{6}-\sqrt{2}}{8}$.

3. $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = k^2 \rightarrow \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} = k^2 \rightarrow \sin x = k^2 - 1$. If $x = \sin^{-1}\frac{1}{3}$, then $\frac{1}{3} = k^2 - 1 \rightarrow k = \frac{2\sqrt{3}}{3}$. Note: k can't be negative since x is a first quadrant angle.

TEAM ROUND Solutions 2007

Team:

1. Since $B^2 - 4C > 0$ then $B > 2\sqrt{C}$. Consider the table below:

C	0	1	2	3	4	5	6	7	8	9
B	1–9	3–9	3–9	4–9	5–9	5–9	5–9	6–9	6–9	7–9
# of B	9	7	7	6	5	5	5	4	4	3

Thus, there are 55 really swell numbers of the form $1BC$.

2. Consider $T = \{1, 2, 3, 4, 5\}$. The sums would go from 3 to 9 with everything in between appearing. Ans: 7. For 1 to n the sums would go from 3 to $n + n - 1 = 2n - 1$. Total $= 2n - 3$. Thus, $2n - 3 > 2007$ so $n > 1005$. Thus, $n = 1006$.

3. Suppose your score is x. If you put in two chips followed by five chips, the score is first $x + 3$, then $2x + 6$. If you put in five chips followed by two chips, the score is first $2x$, then $2x + 3$. Thus, it is better to put in all the possible two-chip moves before any five-chip moves. But how many two-chip moves should there be before you choose the five-chip moves? Here are the possibilities with their point totals:

1. One 2 followed by five 5's: 1, 4, 8, 16, 32, 64, 128
2. Four 2's followed by four 5's: 1, 4, 7, 10, 13, 26, 52, 104, 208
3. Six 2's followed by three 5's: 1, 4, 7, 10, 13, 16, 19, 38, 76, 152
4. Nine 2's followed by two 5's: 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 56, 112
5. Eleven 2's followed by one 5: 1, . . . , 34, 68

The largest score is $\boxed{208}$.

4. In S there are n terms that are divisible by 3. Choosing any three of them gives a sum divisible by 3. There are $n + 1$ terms that are 1(mod 3). Choosing any three of them gives a sum divisible by 3. There are n terms that are 2(mod 3). Choosing any three of them gives a sum divisible by 3. Finally, choosing a number that is divisible by 3, a number equal to 1(mod 3) and a number equal to 2(mod 3) gives a sum divisible by 3. There are $n(n + 1)n$ ways to do this. Thus, a sum divisible by 3 can be obtained in $2\binom{n}{3} + \binom{n+1}{3} + n^2(n + 1)$ equals x ways by choosing 3 elements from set S. In a similar fashion it can be determined that a sum divisible by 3 can be obtained in $3\binom{n}{3} + n^3 = y$ ways from T. Since

$x - y = 330$, we have $2\binom{n}{3} + \binom{n+1}{3} + n^2(n+1) - 3\binom{n}{3} - n^3 = \binom{n+1}{3} - \binom{n}{3} + n^2$

$= \dfrac{3n^2 - n}{2} = 330 \rightarrow 3n^2 - n - 660 = 0 \rightarrow (3n+44)(n-15) = 0$, making $n = 15$.

5. Each side of the polygon cuts off an arc whose degree measure is $\dfrac{360}{n}$. Suppose there are m such arcs in $\overset{\frown}{AB}$, then $m\overset{\frown}{AB} = \dfrac{360m}{n}$. Since $m\angle APB = 33°$, then $m\overset{\frown}{AB} = 147°$ since the angle and arc are supplementary. $147 = \dfrac{360m}{n} \rightarrow n = \dfrac{120}{49}m$. Since n is an integer, $m = 49k$ for k an integer, giving $n = 120k$. Since $n \le 2007$, then $120k \le 2007 \rightarrow$ $k \le 16 + \dfrac{87}{120}$. Thus, there are 16 values for k meaning that there are 16 n-sided regular polygons satisfying the conditions of the problem.

6. From $x^2 + (y-1)^2 = 1$ and $y = ax^2 \rightarrow x^2 = \dfrac{y}{a}$ we have $\dfrac{y}{a} + y^2 - 2y = 0 \rightarrow$ $y\left(y - \dfrac{2a-1}{a}\right) = 0$. This gives two possible solutions: $y = 0$ or $y = \dfrac{2a-1}{a}$. Since $y = 0$ is always a solution, the only way for there to be 1 solution is for $\dfrac{2a-1}{a}$ to be negative when $a > 0$ and positive when $a < 0$. If $a > 0$ then $\dfrac{2a-1}{a} < 0$ for $0 < a < \dfrac{1}{2}$. If $a < 0$, then clearly all values of a work since the parabola is concave down in the 3rd and 4th quadrants while the circle lies above the x-axis except for the origin. If $a = \dfrac{1}{2}$, then both solutions are the same. Answer: $a \le \dfrac{1}{2}$, $a \ne 0$.