MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Round 1 Arithmetic and Number Theory

1. ________________________

2. ________________________

3. ________________________

1. How many integers have a reciprocal that is greater than $\frac{1}{50.1}$ and less than $\frac{1}{\pi}$?

2. Let 9_b, 10_b, and 11_b be numbers in base b. In what positive base b do the numbers form a Pythagorean Triple?

3. Let P_j be the j^{th} prime number. $P_n = 103$ is the n^{th} prime number. $k = P_{n+3} + P_{n+4}$. How many different positive integral factors does k have?

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Round 2 **Algebra 1**

1. ______________________

2. ______________________

3. ______________________

1. Factor completely: $24x^2 + 6xy - 135y^2$

2. Find the largest integer value of x such that $\dfrac{1}{5-\sqrt{x}} < -\dfrac{1}{100}$.

3. Given the system $\begin{cases} ax + by = a^2 \\ bx + ay = b^2 \end{cases}$ with $a \neq \pm b$, find $x - y$ in simplest form.

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STATE PLAYOFFS – 2004

Round 3 – Geometry

1. ________________________

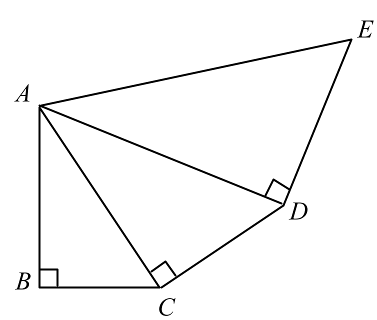
2. ________________________

3. ________________________

1. Each side of square $ABCD$ is of length 12 cm. E is the midpoint of $\overline{AB}$, F is the trisection point of $\overline{BC}$ closer to B and G is on $\overline{CD}$ such that $CG = \frac{1}{4}CD$. How many square centimeters are in the area of ΔEFG?

2. Given $0° < \theta < 90°$, if the ratio of the complement of θ to the supplement of θ is less than one-tenth, determine the number of integer values of θ.

3. $AB = 8$ and $AE = 27$. $\overline{AB} \perp \overline{BC}$, $\overline{AC} \perp \overline{CD}$, and $\overline{AD} \perp \overline{DE}$. Also, $m\angle BAC = m\angle CAD = m\angle DAE$. Find the number of square units in the area of $ABCDE$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Round 4 – Algebra 2

1. ________________________

2. ________________________

3. ________________________

1. $\left(\frac{27}{125}\right)^{-2/3} + \sqrt[4]{\frac{81}{16}}$ can be written in the form $\frac{a}{b}$, where a and b are relatively prime. What is the sum of a and b?

2. If $\log_{16}\left(29 + \log_{16} b\right) = \frac{5}{4}$, determine the value of b.

3. Let a_n be the n^{th} term of an arithmetic progression. Let S_n be the sum of the first n terms of the arithmetic progression with $a_1 = 1$ and $a_3 = 3a_8$. Determine the largest possible value of S_n.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Round 5 – Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Points $A(-10, 0)$, $B(0, 5)$, and $C(10, 0)$ are the vertices of a triangle. How many points $P(x, y)$ are inside the triangle given that x and y are integers.

2. An ellipse has the equation $\frac{(x-2)^2}{4}+\frac{(y)^2}{16}=1$. Find the endpoint in the first quadrant of the chord of the ellipse which passes through its center and is perpendicular to the line whose equation is $x = 2y$. Express your answer as an ordered pair with the coordinates in exact simplified form.

3. Starting at $P(2, 5)$, a bug walks a straight line path, make a 90° turn, and then walks another straight line path until it reaches the point $Q(8, 13)$. Determine the greatest possible distance from the x-axis that the bug can attain.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Round 6 – Trig and Complex Numbers

1. ___________________________

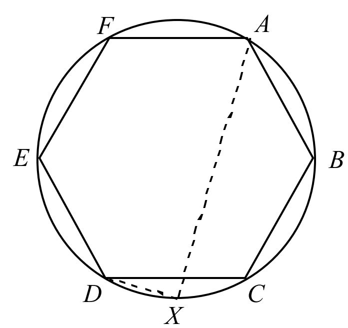
2. ___________________________

3. ___________________________

1. Simplify $(3+4i)^2 - (1-2i)^3$ into the form $a + bi$.

2. If $\sec x - \tan x = 3$, determine the numerical value of $\sec x + \tan x$.

3. Regular hexagon $ABCDEF$ is inscribed in a circle. X is the midpoint of arc $\overset{\frown}{DC}$. Determine the numerical value of $\dfrac{AX}{DX}$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004

Team Round

1. ________________________ **4.** ____________________

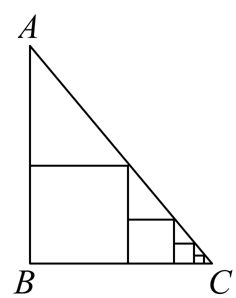
2. ________________________ **5.** ____________________

3. ________________________ **6.** ____________________

1. Let f be a periodic function with a period of 3 defined for all real numbers. If, on the interval (1,4], an equation for $f(x) = y$ is $2x + 3y = 11$ and on the interval (2002, 2005] an equation for $f(x) = y$ is $ax + by = c$ where a, b, and c are relatively prime positive integers, determine the ordered triple (a, b, c).

2. In kite *ABCD*, *AB* = *AD* and *CB* = *CD*. If $m\angle A = 108°$ and $m\angle C = 36°$ then the ratio of the area of ΔABD to the area of ΔCBD can be written in the form $\frac{a - b\tan^2(36°)}{c}$ where a, b, and c are relatively prime positive integers. Determine the ordered triple (a, b, c).

3. An infinite number of squares are inscribed in right ΔABC as indicated in the diagram. If the sum of the areas of the squares is one-fifth the area of ΔABC, determine the ratio of *AB* to *BC*.



4. *ABCD* is an isosceles trapezoid with bases $\overline{AB}$ and $\overline{DC}$ with $AB < DC$. *P* is the point of intersection of $\overline{AC}$ and $\overline{BD}$. Point *X* is chosen at random from the interior of the trapezoid. If the probability that *X* lies in ΔAPD is $\frac{1}{8}$, find $\frac{DC}{AB}$.

5. Determine all values of a such that
$$\ln a^2 + (\ln a^2)^2 + (\ln a^2)^3 + (\ln a^2)^4 + ... = 3(\ln a + (\ln a)^2 + (\ln a)^3 + (\ln a)^4 + ...).$$

6. For $x \in \{1,2,3,\ldots,100\}$, determine the number of positive integer values of y such that $y^2 = x(x+1)(x+2)(x+3)+1$.

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STATE PLAYOFFS – 2004

Answer Sheet

Round 1

1. 47
2. 40
3. 20

Round 2

1. $3(4x-9y)(2x+5y)$
2. 11024
3. $a+b$

Round 3

1. 30
2. 9
3. $133\sqrt{5}$

Round 4

1. 95
2. 4096
3. $\frac{100}{19}$

Round 5

1. 36
2. $\left(2-\sqrt{2},2\sqrt{2}\right)$
3. 14

Round 6

1. $4+22i$
2. $\frac{1}{3}$
3. $2+\sqrt{3}$

Team

1. (2, 3, 4013)
2. $(1,1,2)$
3. 8
4. $3+2\sqrt{2}$
5. 1, $e^{1/4}$
6. 100

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2004 – Solutions

Round 1

1. The numbers from 4 through 50, 47 altogether.

2. $9^2 + (b-0)^2 = (b+1)^2 \rightarrow 81 = 2b+1 \rightarrow b = 40$.

3. $k = 113 + 127 = 240 = 2^4 \cdot 3^1 \cdot 5^1$ which has $5 \cdot 2 \cdot 2 = 20$ factors.

Round 2

1. $3(4x - 9y)(2x + 5y)$

2. $\frac{1}{5-\sqrt{x}} < -\frac{1}{100} \rightarrow 5 - \sqrt{x} > -100 \rightarrow 105 > \sqrt{x} \rightarrow 11025 > x \rightarrow x = 11024.$

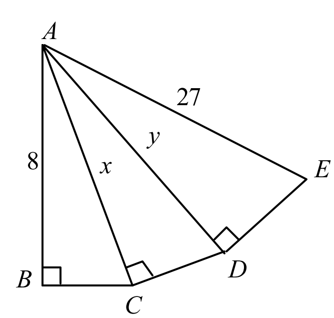
3. Multiplying the top equation by b and the bottom by a and subtracting eliminates x and gives $y = \frac{ab(a-b)}{b^2 - a^2} = \frac{-ab}{a+b}$. Multiplying the top by a and the bottom by b and subtracting eliminates y and gives $x = \frac{a^3 - b^3}{a^2 - b^2} = \frac{a^2 + ab + b^2}{a+b}$.
Thus, $x - y = \frac{a^2 + 2ab + b^2}{a+b} = a + b$.

Round 3

1. Area of square = 144 cm^2. Subtract areas of trapezoid $AEGD$, ΔEBF, and ΔFCG.

2. Since $\frac{90-\theta}{180-\theta} < \frac{1}{10}$ then $900 - 10 \cdot \theta < 180 - \theta \rightarrow \theta > 80$. But since $\theta < 90$, $\theta = 81, 82, \ldots, 89$, making for a total of 9 values of θ.

3. Since the three triangles are similar, $\frac{8}{x} = \frac{x}{y} \rightarrow y = \frac{x^2}{8}$ and $\frac{x}{y} = \frac{y}{27} \rightarrow x = \frac{y^2}{27}$. Thus, $y = \frac{y^4}{8 \cdot 27^2} \rightarrow$ $y^3 = 2^3 \cdot 9^3 \rightarrow y = 18$ and $x = 12$. Then $12^2 - 8^2 = BC^2$ so $BC = 4\sqrt{5}$ and since the scale factor is 3/2, $CD = 6\sqrt{5}$ and $DE = 9\sqrt{5}$. The sum of the areas is $16\sqrt{5} + 36\sqrt{5} + 81\sqrt{5} = 133\sqrt{5}$.



Round 4

1. $\frac{25}{9}+\frac{3}{2}=\frac{77}{18}$; 77 + 18 = 95.

2. $\log_{16}\left(29+\log_{16} b\right) = \frac{5}{4} \rightarrow 29+\log_{16} b = 16^{5/4} = 32 \rightarrow \log_{16} b = 3 \rightarrow b = 16^3$.
Thus, $b = 4096$.

3. From $a_3 = 3a_8$ we obtain $1+2d = 3(1+7d) \rightarrow d = -\frac{2}{19}$. Then
$S_n = \frac{\left(2+(n-1)\left(-\frac{2}{19}\right)\right)n}{2} = \frac{19n - n(n-1)}{19} = \frac{-n^2+20n}{19}$. The maximum occurs at $n = \frac{-20}{-2} = 10$. The maximum of $S_n = \frac{-100+200}{19} = \frac{100}{19}$

Round 5

1 The points are on the horizontal lines y = 1,2,3,4. Considering the slopes of the segments from (0, 5) to (10, 0) and (-10, 0) are –1/2 and 1/2, the points lie between (7, 1) and (-7, 1); (5, 1) and (-5, 1); (3, 3) and –3, 3); and (1, 4) and (-1, 4).

2. The center of the ellipse is (2, 0) and the slope of the chord is –2. The equation of the chord is y = –2x + 4. Substituting into the equation of the ellipse gives $y^2 = 8$, leading to $x = 2-\sqrt{2}$ and $y = 2\sqrt{2}$. The ordered pair is $\left(2-\sqrt{2}, 2\sqrt{2}\right)$

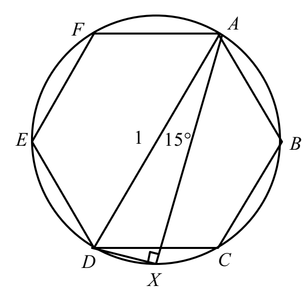
3. Let A be the point at which the bug makes a right hand turn. Then PAQ is a right triangle and can be inscribed in a circle. Thus, all points A lie on a circle with diameter $\overline{PQ}$. The circle's diameter equals $\sqrt{(8-2)^2+(13-5)^2} = 10$, the circle's center is the midpoint of $\overline{PQ}$, namely (5, 9), making the circle's equation equal $(x-5)^2+(y-9)^2 = 25$. The high point occurs directly above the center where $x = 5$ making $y = 14$.

Round 6

1. $(3+4i)^2 = -7+24i$; $(1-2i)^3 = -11+2i$; $-7+24i+11-2i = 4+22i$.

2. $(\sec x - \tan x)(\sec x + \tan x) = 3(\sec x + \tan x) \rightarrow \sec^2 x - \tan^2 x = 3(\sec x + \tan x)$
giving $1 = 3(\sec x + \tan x)$ so $\sec x + \tan x = \frac{1}{3}$.

3. Since $\overline{AD}$ is a diameter, AXD is a right triangle and since $m\widehat{DX} = 30°$, then $m\angle DAX = 15°$. Set $AD = 1$, then $\frac{AX}{DX} = \frac{\sin 75°}{\sin 15°} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = 2+\sqrt{3}$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

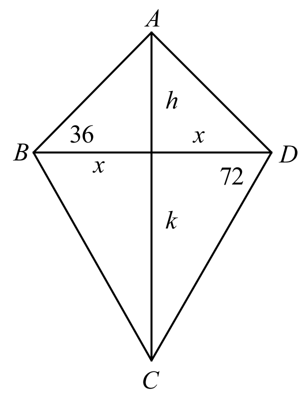
STATE PLAYOFFS – 2001 – Solutions – Team Round

1. On [1, 4] the linear function joins (1, 3) and (4, 1), so on [2002, 2005] the linear function will connect (2002, 3) and (2005, 1). The equation is $y-1=-\frac{2}{3}(x-2005)$ giving $2x+3y=4013$ so the answer is (2, 3, 4013).

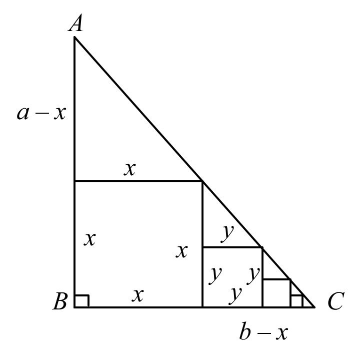
2. Since the triangles have a common base, the ratio of their areas equals the ratio of their heights. Since $\tan 36=\frac{h}{x}$, then $h=x\tan 36$ Since $\tan 72=\frac{k}{x}$ then $k=x\tan 72$.

Hence, $\frac{h}{k}=\frac{x\tan 36}{x\tan 72}=\frac{\tan 36}{\frac{2\tan 36}{1-\tan^2 36}}=\frac{1-\tan^2 36}{2}$.

The answer is $(1,1,2)$



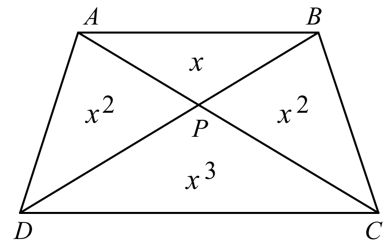
3. Let x be the side of the first square and y be the side of the second square. Since $\frac{a-x}{x}=\frac{a}{b}$ then $x=\frac{ab}{a+b}$.
Applying the same approach to the square with side y inscribed in a triangle with legs x and $b-x$, then $y=\frac{x(b-x)}{x+(b-x)}=\frac{x(b-x)}{b}=\frac{ab}{a+b}\left(b-\frac{ab}{a+b}\right)\cdot\frac{1}{b}=\left(\frac{ab}{a+b}\right)\left(\frac{b}{a+b}\right)$. Hence $\frac{b}{a+b}$ is the common ratio of the sides. The sum of the areas equals



$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{ab}{a+b}\right)^2\left(\frac{b}{a+b}\right)^2+\left(\frac{ab}{a+b}\right)^2\left(\frac{b}{a+b}\right)^4+K=\frac{\left(\frac{ab}{a+b}\right)^2}{1-\frac{b^2}{(a+b)^2}}=\frac{ab^2}{a+2b}.$$ Thus,

$$\frac{1}{5}\text{area }\Delta ABC=\frac{1}{5}\left(\frac{1}{2}ab\right)=\frac{ab^2}{a+2b}\rightarrow\frac{b}{a+2b}=\frac{1}{10}\text{ gives }\frac{a}{b}=8.$$

4. It is helpful to know that the areas of ΔPBC and ΔPAD are equal and that the areas of ΔPAB, ΔPBC, and ΔPCD form an increasing geometric progression. Let the area of $\Delta PAB = x$, then the other areas are as marked on the diagram. Thus, $\dfrac{2x^2}{x+2x^2+x^3} = \dfrac{1}{4}$ gives $x^2 - 6x + 1 = 0$.



Since $\dfrac{DC}{AB} = \sqrt{\dfrac{x^3}{x}} = x$, we solve the equation and obtain $x = 3 + 2\sqrt{2}$.

5. $\dfrac{\ln a^2}{1-\ln a^2} = \dfrac{3\ln a}{1-\ln a} \rightarrow \dfrac{2\ln a}{1-2\ln a} = \dfrac{3\ln a}{1-\ln a} \rightarrow 2(\ln a) - 2(\ln a)^2 = 3\ln a - 6(\ln a)^2$
$\rightarrow 4(\ln a)^2 - \ln a = 0 \rightarrow \ln a = 0 \ or \ \dfrac{1}{4}$. Thus, $a = 1, e^{1/4}$.

6. Subtract 1 from both sides obtaining: $(y-1)(y+1) = x(x+1)(x+2)(x+3) =$ $(x(x+3))((x+1)(x+2)) = (x^2+3x)(x^2+3x+2)$. Since $y-1$ and $y+1$ are two integers differing by 2 and x^2+3x and x^2+3x+2 are two integers differing by 2, then for each value of x in $\{1, 2, \ldots, 100\}$ there is a value of y for which $(y-1)(y+1) = x(x+1)(x+2)(x+3)$. For example, when $x = 1$, $y = 5$; when $x = 100$, $y = 10{,}301$. The answer is 100.